Challenges in evaluating costs of known lattice attacks

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Based on attack survey from 2019 Bernstein–Chuengsatiansup–Lange–van Vredendaal.

Why analysis is important:

- Guide attack optimization.
- Guide attack selection.
- Evaluate crypto parameters.
- Evaluate crypto designs.
- Advise users on security.
Three typical attack problems

Define $\mathcal{R} = \mathbb{Z}[x]/(x^{761} - x - 1)$; “small” = all coeffs in $\{-1, 0, 1\}$; $w = 286$; $q = 4591$.

Attacker wants to find small weight-$w$ secret $a \in \mathcal{R}$.

Problem 1: Public $G \in \mathcal{R}/q$ with $aG + e = 0$. Small secret $e \in \mathcal{R}$.

Problem 2: Public $G \in \mathcal{R}/q$ and $aG + e$. Small secret $e \in \mathcal{R}$.

Problem 3: Public $G_1, G_2 \in \mathcal{R}/q$. Public $aG_1 + e_1, aG_2 + e_2$. Small secrets $e_1, e_2 \in \mathcal{R}$. 
Examples of target cryptosystems

Secret key: small $a$; small $e$.

Public key reveals multiplier $G$ and approximation $A = aG + e$.

Public key for “NTRU”: $G = -e/a$, and $A = 0$. 
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Systematization of naming, recognizing similarity + credits: “NTRU” \( \Rightarrow \) Quotient NTRU. “Ring-LWE” \( \Rightarrow \) Product NTRU.
Encryption for Quotient NTRU:
Input small $b$, small $d$.
Ciphertext: $B = 3Gb + d$. 
Encryption for Quotient NTRU:  
Input small \( b \), small \( d \).  
Ciphertext: \( B = 3Gb + d \).

Encryption for Product NTRU:  
Input encoded message \( M \).  
Randomly generate  
small \( b \), small \( d \), small \( c \).  
Ciphertext: \( B = Gb + d \)  
and \( C = Ab + M + c \).
Encryption for Quotient NTRU:
Input small $b$, small $d$.
Ciphertext: $B = 3G\cdot b + d$.

Encryption for Product NTRU:
Input encoded message $M$.
Randomly generate small $b$, small $d$, small $c$.
Ciphertext: $B = G\cdot b + d$
and $C = A\cdot b + M + c$.

Next slides: survey of $G, a, e, c, M$ details and variants in NISTPQC submissions. Source: Bernstein, “Comparing proofs of security for lattice-based encryption”. 
<table>
<thead>
<tr>
<th>system</th>
<th>parameter set</th>
<th>type</th>
<th>set of multipliers</th>
</tr>
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<tbody>
<tr>
<td>frodo</td>
<td>640</td>
<td>Product</td>
<td>$(\mathbb{Z}/32768)^{640 \times 640}$</td>
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<tr>
<td>frodo</td>
<td>976</td>
<td>Product</td>
<td>$(\mathbb{Z}/65536)^{976 \times 976}$</td>
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<td>frodo</td>
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<td>512</td>
<td>Product</td>
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<td>768</td>
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<td>$((\mathbb{Z}/3329)[x]/(x^{256} + 1))^{3 \times 3}$</td>
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<tr>
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<td>Product</td>
<td>$((\mathbb{Z}/3329)[x]/(x^{256} + 1))^{4 \times 4}$</td>
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<td>lac</td>
<td>128</td>
<td>Product</td>
<td>$(\mathbb{Z}/251)[x]/(x^{512} + 1)$</td>
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<tr>
<td>lac</td>
<td>192</td>
<td>Product</td>
<td>$(\mathbb{Z}/251)[x]/(x^{1024} + 1)$</td>
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<tr>
<td>lac</td>
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<tr>
<td>newhope</td>
<td>512</td>
<td>Product</td>
<td>$(\mathbb{Z}/12289)[x]/(x^{512} + 1)$</td>
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<tr>
<td>newhope</td>
<td>1024</td>
<td>Product</td>
<td>$(\mathbb{Z}/12289)[x]/(x^{1024} + 1)$</td>
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<tr>
<td>ntru</td>
<td>hps2048509</td>
<td>Quotient</td>
<td>$(\mathbb{Z}/2048)[x]/(x^{509} - 1)$</td>
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<tr>
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<td>hrss701</td>
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<td>653</td>
<td>Product</td>
<td>$(\mathbb{Z}/4621)[x]/(x^{653} - x - 1)$</td>
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<td>ntrulpr</td>
<td>761</td>
<td>Product</td>
<td>$(\mathbb{Z}/4591)[x]/(x^{761} - x - 1)$</td>
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<tr>
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<td>Product</td>
<td>$(\mathbb{Z}/5167)[x]/(x^{857} - x - 1)$</td>
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<td>Product</td>
<td>$(\mathbb{Z}/4096)^{636 \times 636}$</td>
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<td>round5n1</td>
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<td>Product</td>
<td>$(\mathbb{Z}/32768)^{876 \times 876}$</td>
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<td>round5n1</td>
<td>5</td>
<td>Product</td>
<td>$(\mathbb{Z}/32768)^{1217 \times 1217}$</td>
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<tr>
<td>round5nd</td>
<td>1.0d</td>
<td>Product</td>
<td>$(\mathbb{Z}/8192)[x]/(x^{586} + \ldots + 1)$</td>
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<td>Product</td>
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<tr>
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<td>Product</td>
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<td>5.5d</td>
<td>Product</td>
<td>$(\mathbb{Z}/2048)[x]/(x^{947} - 1)$</td>
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<td>saber</td>
<td>light</td>
<td>Product</td>
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<td>main</td>
<td>Product</td>
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</tr>
<tr>
<td>saber</td>
<td>fire</td>
<td>Product</td>
<td>$((\mathbb{Z}/8192)[x]/(x^{256} + 1))^{4 \times 4}$</td>
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<td>sntrup</td>
<td>653</td>
<td>Quotient</td>
<td>$(\mathbb{Z}/4621)[x]/(x^{653} - x - 1)$</td>
</tr>
<tr>
<td>sntrup</td>
<td>761</td>
<td>Quotient</td>
<td>$(\mathbb{Z}/4591)[x]/(x^{761} - x - 1)$</td>
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<tr>
<td>sntrup</td>
<td>857</td>
<td>Quotient</td>
<td>$(\mathbb{Z}/5167)[x]/(x^{857} - x - 1)$</td>
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<tr>
<td>threebears</td>
<td>baby</td>
<td>Product</td>
<td>$(\mathbb{Z}/(2^{3120} - 2^{1560} - 1))^{2 \times 2}$</td>
</tr>
<tr>
<td>threebears</td>
<td>mama</td>
<td>Product</td>
<td>$(\mathbb{Z}/(2^{3120} - 2^{1560} - 1))^{3 \times 3}$</td>
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<tr>
<td>threebears</td>
<td>papa</td>
<td>Product</td>
<td>$(\mathbb{Z}/(2^{3120} - 2^{1560} - 1))^{4 \times 4}$</td>
</tr>
</tbody>
</table>
short element

$\mathbb{Z}^{640 \times 8}$; \{-12, \ldots, 12\}; Pr 1, 4, 17, \ldots (spec page 23)

$\mathbb{Z}^{976 \times 8}$; \{-10, \ldots, 10\}; Pr 1, 6, 29, \ldots (spec page 23)

$\mathbb{Z}^{1344 \times 8}$; \{-6, \ldots, 6\}; Pr 2, 40, 364, \ldots (spec page 23)

$(\mathbb{Z}[x]/(x^{256} + 1))^2$; $\sum_{0 \leq i < 4}$ \{-0.5, 0.5\}

$(\mathbb{Z}[x]/(x^{256} + 1))^3$; $\sum_{0 \leq i < 4}$ \{-0.5, 0.5\}

$(\mathbb{Z}[x]/(x^{256} + 1))^4$; $\sum_{0 \leq i < 4}$ \{-0.5, 0.5\}

$(\mathbb{Z}[x]/(x^{512} + 1))^2$; $\sum_{0 \leq i < 16}$ \{-0.5, 0.5\}

$(\mathbb{Z}[x]/(x^{1024} + 1))^2$; $\sum_{0 \leq i < 16}$ \{-0.5, 0.5\}

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; Pr 1, 2, 1; weight 128, 128

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; Pr 1, 6, 1; weight 128, 128

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; Pr 1, 2, 1; weight 256, 256

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; key correlation $\geq 0$

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 252

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 250

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 281

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 57, 57

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 223, 223

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 231, 231

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 91, 91

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 106, 106

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 111, 111

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 68, 68; ending 0

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 121, 121; ending 0

$\mathbb{Z}^{625 \times 1}$; \{-1, 0, 1\}; weight 194, 194; ending 0

$(\mathbb{Z}[x]/(x^{256} + 1))^2$; $\sum_{0 \leq i < 10}$ \{-0.5, 0.5\}

$(\mathbb{Z}[x]/(x^{256} + 1))^3$; $\sum_{0 \leq i < 8}$ \{-0.5, 0.5\}

$(\mathbb{Z}[x]/(x^{256} + 1))^4$; $\sum_{0 \leq i < 6}$ \{-0.5, 0.5\}

$\mathbb{Z}^{640 \times 8}$; \{-12, \ldots, 12\}; Pr 1, 32, 62, 32, 1; *

$\mathbb{Z}^{640 \times 8}$; \{-10, \ldots, 10\}; Pr 13, 38, 13; *

$\mathbb{Z}^{640 \times 8}$; \{-6, \ldots, 6\}; Pr 5, 22, 5; *
key offset (numerator or noise or rounding method)

\[ Z^{640 \times 8}; \{ -12, \ldots, 12 \}; \Pr 1, 4, 17, \ldots \] (spec page 23)

\[ Z^{976 \times 8}; \{ -10, \ldots, 10 \}; \Pr 1, 6, 29, \ldots \] (spec page 23)

\[ Z^{1344 \times 8}; \{ -6, \ldots, 6 \}; \Pr 2, 40, 364, \ldots \] (spec page 23)

\( Z \times (x^{256} + 1)^2; \sum_{0 \leq i < 4} \{-0.5, 0.5\} \)

\( Z \times (x^{256} + 1)^3; \sum_{0 \leq i < 4} \{-0.5, 0.5\} \)

\( Z \times (x^{256} + 1)^4; \sum_{0 \leq i < 4} \{-0.5, 0.5\} \)

\( Z \times (x^{256} + 1); \{-1, 0, 1\}; \Pr 1, 2, 1; \text{ weight } 128, 128 \)

\( Z \times (x^{1024} + 1); \{-1, 0, 1\}; \Pr 1, 6, 1; \text{ weight } 128, 128 \)

\( Z \times (x^{1024} + 1); \{-1, 0, 1\}; \Pr 1, 2, 1; \text{ weight } 256, 256 \)

\( Z \times (x^{512} + 1); \sum_{0 \leq i < 16} \{-0.5, 0.5\} \)

\( Z \times (x^{1024} + 1); \sum_{0 \leq i < 16} \{-0.5, 0.5\} \)

\( Z \times (x^{509} - 1); \{-1, 0, 1\}; \text{ weight } 127, 127 \)

\( Z \times (x^{677} - 1); \{-1, 0, 1\}; \text{ weight } 127, 127 \)

\( Z \times (x^{821} - 1); \{-1, 0, 1\}; \text{ weight } 255, 255 \)

\( Z \times (x^{701} - 1); \{-1, 0, 1\}; \text{ key correlation } \geq 0; \cdot (x - 1) \)

round \( \{ -2310, \ldots, 2310 \} \) to \( 3Z \)

round \( \{ -2295, \ldots, 2295 \} \) to \( 3Z \)

round \( \{ -2583, \ldots, 2583 \} \) to \( 3Z \)

round \( Z \div 4096 \) to \( 8Z \)

round \( Z \div 32768 \) to \( 16Z \)

round \( Z \div 32768 \) to \( 8Z \)

round \( Z \div 8192 \) to \( 16Z \)

round \( Z \div 4096 \) to \( 8Z \)

round \( Z \div 8192 \) to \( 16Z \)

reduce mod \( x^{508} + \ldots + 1 \); round \( Z \div 1024 \) to \( 8Z \)

reduce mod \( x^{756} + \ldots + 1 \); round \( Z \div 4096 \) to \( 16Z \)

reduce mod \( x^{946} + \ldots + 1 \); round \( Z \div 2048 \) to \( 8Z \)

round \( Z \div 8192 \) to \( 8Z \)

round \( Z \div 8192 \) to \( 8Z \)

round \( Z \div 8192 \) to \( 8Z \)

\( Z \times (x^{653} - x - 1); \{-1, 0, 1\}; \text{ invertible mod } 3 \)

\( Z \times (x^{761} - x - 1); \{-1, 0, 1\}; \text{ invertible mod } 3 \)

\( Z \times (x^{857} - x - 1); \{-1, 0, 1\}; \text{ invertible mod } 3 \)

\( Z^2; \sum_{0 \leq i < 312} 2^{10i}; \{ -2, -1, 0, 1, 2 \}; \Pr 13, 32, 62, 32, 1; * \)

\( Z^3; \sum_{0 \leq i < 312} 2^{10i}; \{ -1, 0, 1 \}; \Pr 13, 38, 13; * \)

\( Z^4; \sum_{0 \leq i < 312} 2^{10i}; \{ -1, 0, 1 \}; \Pr 5, 22, 5; * \)
ciphertext offset (noise or rounding method)

\[ Z^{8 \times 8}, \{ -12, \ldots, 12 \}; \text{ Pr 1, 4, 17, \ldots} \] (spec page 23)
\[ Z^{8 \times 8}, \{ -10, \ldots, 10 \}; \text{ Pr 1, 6, 29, \ldots} \] (spec page 23)
\[ Z^{8 \times 8}, \{ -6, \ldots, 6 \}; \text{ Pr 2, 40, 364, \ldots} \] (spec page 23)

\[ Z[x]/(x^{256} + 1); \sum_{0 \leq i < 4} \{-0.5, 0.5\} \]
\[ Z[x]/(x^{256} + 1); \sum_{0 \leq i < 4} \{-0.5, 0.5\} \]
\[ Z[x]/(x^{512} + 1); \{-1, 0, 1\}; \text{ Pr 1, 2, 1} \]
\[ Z[x]/(x^{1024} + 1); \{-1, 0, 1\}; \text{ Pr 1, 6, 1} \]
\[ Z[x]/(x^{1024} + 1); \{-1, 0, 1\}; \text{ Pr 1, 2, 1} \]
\[ Z[x]/(x^{512} + 1); \sum_{0 \leq i < 16} \{-0.5, 0.5\} \]
\[ Z[x]/(x^{1024} + 1); \sum_{0 \leq i < 16} \{-0.5, 0.5\} \]

not applicable
not applicable
not applicable
not applicable

bottom 256 coeffs; \( z \mapsto \left\lfloor \frac{(114(z + 2156) + 16384)}{32768} \right\rfloor \)
bottom 256 coeffs; \( z \mapsto \left\lfloor \frac{(113(z + 2175) + 16384)}{32768} \right\rfloor \)
bottom 256 coeffs; \( z \mapsto \left\lfloor \frac{(101(z + 2433) + 16384)}{32768} \right\rfloor \)

round \( Z/4096 \) to \( 64Z \)
round \( Z/32768 \) to \( 512Z \)
round \( Z/32768 \) to \( 64Z \)

bottom 128 coeffs; round \( Z/8192 \) to \( 512Z \)
bottom 192 coeffs; round \( Z/4096 \) to \( 128Z \)
bottom 256 coeffs; round \( Z/8192 \) to \( 256Z \)
bottom 318 coeffs; round \( Z/1024 \) to \( 64Z \)
bottom 410 coeffs; round \( Z/4096 \) to \( 512Z \)
bottom 490 coeffs; round \( Z/2048 \) to \( 64Z \)
round \( Z/8192 \) to \( 1024Z \)
round \( Z/8192 \) to \( 512Z \)
round \( Z/8192 \) to \( 128Z \)
not applicable
not applicable
not applicable
not applicable

\( Z; \sum_{0 \leq i < 312} 2^{10i} \{-2, -1, 0, 1, 2\}; \text{ Pr 1, 32, 62, 32, 1;} \)*
\( Z; \sum_{0 \leq i < 312} 2^{10i} \{-1, 0, 1\}; \text{ Pr 13, 38, 13;} \)*
\( Z; \sum_{0 \leq i < 312} 2^{10i} \{-1, 0, 1\}; \text{ Pr 5, 22, 5;} \)*
set of encoded messages

8 × 8 matrix over \{0, 8192, 16384, 24576\}
8 × 8 matrix over \{0, 8192, \ldots, 57344\}
8 × 8 matrix over \{0, 4096, \ldots, 61440\}
\begin{align*}
\sum_{0 \leq i < 256} & \{0, 1665\} x^i \\
\sum_{0 \leq i < 256} & \{0, 1665\} x^i \\
\sum_{0 \leq i < 256} & \{0, 1665\} x^i \\
\end{align*}

256-dim subcode (see spec) of \begin{align*}
\sum_{0 \leq i < 512} & \{0, 126\} x^i \\
\sum_{0 \leq i < 1024} & \{0, 126\} x^i \\
\sum_{0 \leq i < 1024} & \{0, 126\} x^i \\
\end{align*}

\begin{align*}
\sum_{0 \leq i < 256} & \{0, 6145\} x^i (1 + x^{256}) \\
\sum_{0 \leq i < 256} & \{0, 6145\} x^i (1 + x^{256} + x^{512} + x^{768}) \\
\end{align*}

not applicable

not applicable

not applicable

\begin{align*}
\sum_{0 \leq i < 256} & \{0, 2310\} x^i \\
\sum_{0 \leq i < 256} & \{0, 2295\} x^i \\
\sum_{0 \leq i < 256} & \{0, 2583\} x^i \\
\end{align*}

8 × 8 matrix over \{0, 1024, 2048, 3072\}
8 × 8 matrix over \{0, 4096, \ldots, 28672\}
8 × 8 matrix over \{0, 2048, \ldots, 30720\}
\begin{align*}
\sum_{0 \leq i < 128} & \{0, 4096\} x^i \\
\sum_{0 \leq i < 192} & \{0, 2048\} x^i \\
\sum_{0 \leq i < 256} & \{0, 4096\} x^i \\
\end{align*}

128-dim subcode (see spec) of \begin{align*}
\sum_{0 \leq i < 318} & \{0, 512\} x^i \\
\sum_{0 \leq i < 410} & \{0, 2048\} x^i \\
\sum_{0 \leq i < 490} & \{0, 1024\} x^i \\
\end{align*}

\begin{align*}
\sum_{0 \leq i < 256} & \{0, 4096\} x^i \\
\sum_{0 \leq i < 256} & \{0, 4096\} x^i \\
\sum_{0 \leq i < 256} & \{0, 4096\} x^i \\
\end{align*}

not applicable

not applicable

not applicable

\begin{align*}
\sum_{0 \leq i < 274} & \{0, 512\} 2^{10i} \\
\sum_{0 \leq i < 274} & \{0, 512\} 2^{10i} \\
\sum_{0 \leq i < 274} & \{0, 512\} 2^{10i} \\
\end{align*}
Attacking these problems

Attack strategy with reputation of usually being best: “primal” strategy. Focus of this talk.

Normal layers in analysis:

Analysis of lattices to attack systems

“Approximate-SVP” analysis

“SVP” analysis

Model of computation
Models of computation

Multitape Turing machine: e.g., sort $N$ ints, each $N^{o(1)}$ bits, in time $N^{1+o(1)}$, space $N^{1+o(1)}$. 
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Brent–Kung 2D circuit model allows parallelism—e.g., sort in time $N^{0.5+o(1)}$, space $N^{1+o(1)}$. 

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PRAM: multiple inequivalent definitions, untethered to physical explanations. Sort in time $N^{o(1)}$. 
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Multitape Turing machine: e.g., sort $N$ ints, each $N^{o(1)}$ bits, in time $N^{1+o(1)}$, space $N^{1+o(1)}$.

Brent–Kung 2D circuit model allows parallelism—e.g., sort in time $N^{0.5+o(1)}$, space $N^{1+o(1)}$.

PRAM: multiple inequivalent definitions, untethered to physical explanations. Sort in time $N^{o(1)}$.

Quantum computing: similar divergence of models.
Lattices

Rewrite each problem as finding **short** nonzero solution to system of homogeneous $\mathcal{R}/q$ equations.

Problem 1: Find $(a, e) \in \mathcal{R}^2$ with $aG + e = 0$, given $G \in \mathcal{R}/q$. 
Lattices

Rewrite each problem as finding \textbf{short} nonzero solution to system of homogeneous \( \mathcal{R}/q \) equations.

Problem 1: Find \((a, e) \in \mathcal{R}^2\) with \(aG + e = 0\), given \(G \in \mathcal{R}/q\).

Problem 2: Find \((a, t, e) \in \mathcal{R}^3\) with \(aG + e = At\), given \(G, A \in \mathcal{R}/q\).
Lattices

Rewrite each problem as finding **short** nonzero solution to system of homogeneous $\mathcal{R}/q$ equations.

**Problem 1:** Find $(a, e) \in \mathcal{R}^2$ with $aG + e = 0$, given $G \in \mathcal{R}/q$.

**Problem 2:** Find $(a, t, e) \in \mathcal{R}^3$ with $aG + e = At$, given $G, A \in \mathcal{R}/q$.

**Problem 3:** Find $(a, t_1, t_2, e_1, e_2) \in \mathcal{R}^5$ with $aG_1 + e_1 = A_1 t_1$, $aG_2 + e_2 = A_2 t_2$, given $G_1, A_1, G_2, A_2 \in \mathcal{R}/q$. 
Recognize each solution space as a full-rank lattice:

Problem 1: Lattice is image of the map \((\bar{a}, \bar{r}) \mapsto (\bar{a}, q\bar{r} - \bar{a}G)\) from \(\mathcal{R}^2\) to \(\mathcal{R}^2\).
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Module structure

Each of these lattices is an $R$-module, and thus has, generically, many independent short vectors.
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e.g. in Problem 2:
Lattice has short $(a, t, e)$.
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Many more lattice vectors are fairly short combinations of independent vectors:
e.g., $((x + 1)a, (x + 1)t, (x + 1)e)$.  

2001 May–Silverman, for Problem 1: Force a few coefficients of $a$ to be 0. This reduces lattice rank, speeding up various attacks, despite lower success chance.
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Other problems: same speedup. e.g. Problem 2: Force many coefficients of \((a, t)\) to be 0. Bai–Galbraith special case: Force \( t = 1 \), and force a few coefficients of \( a \) to be 0.

(Also slowdown if \( q \) is very large?)
Standard analysis for Problem 1

Lattice has rank $2 \cdot 761 = 1522$.

Uniform random small weight-$w$ secret $a$ has length $\sqrt{w} \approx 17$. 
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Attack parameter: $k = 13$.

Force $k$ positions in $a$ to be 0: restrict to sublattice of rank 1509.

$\Pr[a \text{ is in sublattice}] \approx 0.2\%$. 
Attacker is just as happy to find another solution such as \((xa, xe)\).
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Pretend this analysis applies to \(\mathbb{Z}[x]/(x^{761} - x - 1)\). (It doesn’t.)
Write equation $e = qr - aG$
as 761 equations on coefficients.
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Attack parameter: $m = 600$.

Ignore $761 - m = 161$ equations: i.e., project $e$ onto 600 positions.

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Projected sublattice rank $d = 1509 - 161 = 1348$; det $q^{600}$.

Attack parameter: $\lambda = 1.331876$.

Rescaling: Assign weight $\lambda$ to positions in $a$. Increases length of $a$ to $\lambda \sqrt{w} \approx 23$; increases det to $\lambda^{748} q^{600}$. (Is this $\lambda$ optimal? Interaction with $e$ size variation?)
Lattice-basis reduction

Attack parameter: $\beta = 525$.

Use BKZ-$\beta$ algorithm to reduce lattice basis. (What about alternatives to BKZ?)
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Standard analysis of BKZ-$\beta$:

“Normally” finds nonzero vector of length $\delta^d (\det L)^{1/d}$ where

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(This $\delta$ formula is an asymptotic claim without claimed error bounds. Does not match experiments for specific $d$.)
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Hence the attack finds \((a, e)\), assuming forcing worked. If it didn’t, retry. (Are these tries independent? Should they use new parameters? Grover?)
How long does BKZ-\(\beta\) take?

Standard answer: \(2^{0.292\beta} = 2^{153.3}\) operations by “sieving”.
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\(0.292\beta\) (fake) cost for “sieving” is advertised as being below \(0.187\beta \log_2 \beta - 1.019\beta + 16.1\) (questionable extrapolation of experiments) for “enumeration”.)
Note fragility of comparison.

\[ S \leq 43 \Rightarrow E < S \text{ for } S = 0.396 \beta, \quad E = 0.187 \beta \log_2 \beta - 1.019 \beta + 16.1. \]
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Need to get analyses right!
First step: include models that account for memory cost.
sntrup761 evaluations from “NTRU Prime: round 2” Table 2:

**Ignoring hybrid attacks:**

<table>
<thead>
<tr>
<th>Enum, Free Memory Cost</th>
<th>Enum, Real Memory Cost</th>
<th>Sieving, Free Memory Cost</th>
<th>Sieving, Real Memory Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>368</td>
<td>185</td>
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**Security levels:**

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<tr>
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e.g. Problem 1: $aG$ small so $a_1G \approx -a_2G$. (How fast are near-neighbor algorithms?)
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Common claim: This saves time only for sufficiently narrow $\{a\}$. (Is this true, or a calculation error in existing algorithm analyses?)