

Explanation:

Miller Rabin: How to apply?

Check primality of n where $n-1=2^r \cdot t$, t odd

1. Pick random $a > 0$
2. Compute $b \equiv a^t \pmod n$ (congruence)
3. If $b \in \{-1, 1\}$ then "probably prime"
4. For $i=1$ to $r-1$ do
 - a) compute $b \equiv b^2 \pmod n$ (assigning to b the new value)
 - b) if $b \equiv -1$ output "probably prime"
 - c) if $b \equiv 1$ output "n not prime"
5. output "n not prime"

Iterate this for l choices of a to get probability of 2^{l-1} .

Why does it work?

Fermat says $a^{(n-1)} \equiv a^{(t \cdot 2^r)} \equiv 1 \pmod n$ if n is prime

so in the final squaring we need to reach 1 or n is not prime.

If n is prime then there are 2 square roots of 1, namely 1 and -1.

If $n = p \cdot q$ then there are 4 roots, for k different factors there are 2^k roots, because of CRT.

$$x^2 \equiv 1 \pmod n, \text{ let } n = p \cdot q.$$

$$x^2 \equiv 1 \pmod p$$

$$x^2 \equiv 1 \pmod q$$

p and q are primes, so there are 2 square roots. This gives 4 different CRT systems

$$x \equiv \pm 1 \pmod p$$

$$x \equiv \pm 1 \pmod q$$

with signs taken independently, these give 4 different solutions, namely

$x \equiv 1 \pmod n$ for both choices $+$, $x \equiv -1 \pmod n$ if both choices are $-$ and a different solution $x \equiv c \pmod n$ in the case of $+$ for p , $-$ for q and $-c$ in the other.

If we find c with $c^2 \equiv 1 \pmod n$ and c is not ± 1 then n cannot be prime.

Miller Rabin tries to find such c , knowing that $a^{(n-1)} \equiv 1$, and we can compute r square roots of that -- by building the powers of a^t by squaring. If Fermat holds, we must compute 1 eventually, and if n is prime, we must have encountered -1 before that.

This covers the for loop, if a^t is already 1 then we don't get any information.