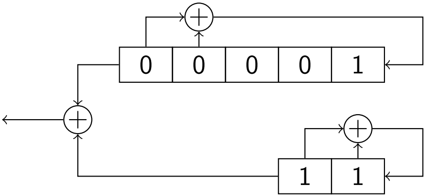
LFSRs: Math vs. mystery

Tanja Lange

Eindhoven University of Technology

2WF80: Introduction to Cryptology

A fourth example



These LFSRs produce $\overline{000010001100101011111}$ and $\overline{011}$ of periods 21 and 3.

Their sum gives

```
  0 0 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 1 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
---------------------------------------------
  0 1 1 0 0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0
```

of period 21.

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---------------------------------------------
  1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1
```

of period 7?

Our hypotheses would have predicted: 21, 21, 21, 21, 3, 1 and some more for the $2^5 - 21 - 1 = 10$ missing states in the first. But we do not get the fourth 21.

Some notation

- Given an LFSR with state size n, characteristic polynomial $P(x)$.
- For a polynomial $f(x)$ denote by $f^*(x)$ its reciprocal

$$f^*(x) = \left(\sum_{i=0}^{n} f_i x^i\right)^* = x^n \sum_{i=0}^{n} f_i x^{-i} = \sum_{i=0}^{n} f_i x^{n-i} = \sum_{i=0}^{n} f_{n-i} x^i.$$

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- The generating function of a sequence $\{s_i\}_i$ is given by

$$S(x) = \sum_{i=0}^{\infty} s_i x^i.$$

Note: S depends on the starting state; there are 2^n different generating functions for an LFSR with state size n.

Some notation and helpful results

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Claim: $\deg(P^*(x)S(x)) < n$

Simplify notation: put $c_n = 1$

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Characterization of characteristic polynomial

This gives an alternative definition of the characteristic polynomial:

Lemma

Let $F(x)$ of $\deg(F) < n$ and $P(x) = x^n + \sum_{i=0}^{n-1} c_i x^i$ with $c_0 = 1$. Then the power series

$$S(x) = F(x)/P^*(x)$$

is the generating function of an LFSR with state size n satisfying $s_{k+n} = \sum_{j=0}^{n-1} c_j s_{k+j}$.

Proof computes $P^*(x)S(x)$.
Then observes that $\deg(F) < n$ forces cancellations as in previous proof.

A promised proof

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Lemma

Let $P(x)$ with $\deg(P) = n$ be the characteristic polynomial of an LFSR. If $P(x)$ is irreducible and has order ℓ then all non-zero starting states give sequences of period ℓ.

Proof.

Let $\{s_i\}_i$ have period r.

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Put $\bar{S}(x) = \sum_{i=0}^{r-1} s_i x^i$. Then $S(x) = \bar{S}(x)\left(1 + x^r + x^{2r} + \cdots\right)$.

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Together this gives $r = \ell$. □

Theorem

Let $\{s_i\}_i$ and $\{t_i\}_i$ be sequences from LFSRs with characteristic polynomials $P(x)$ and $Q(x)$.
There exists an LFSR with output matching $\{s_i + t_i\}_i$.
Its characteristic polynomial is $\mathrm{lcm}(P(x), Q(x))$.

Proof.

The generating function of the sum is

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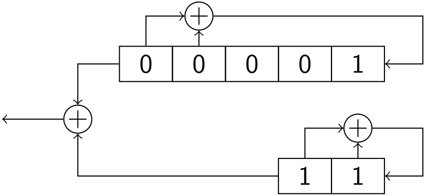
$$\deg(a(x)F(x) + b(x)G(x)) < \deg(R)$$

as $\deg(F) < \deg(P)$ and $\deg(G) < \deg(Q)$.
All this holds independent of the starting states. □

A mystery solved



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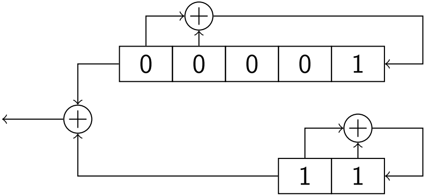

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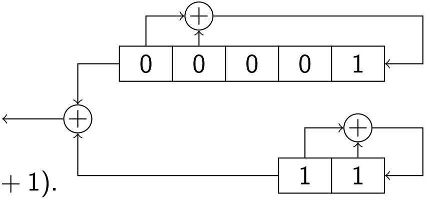

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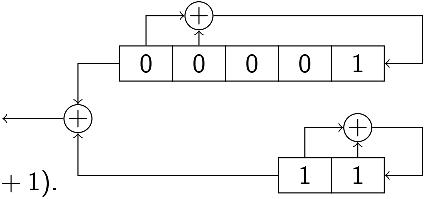

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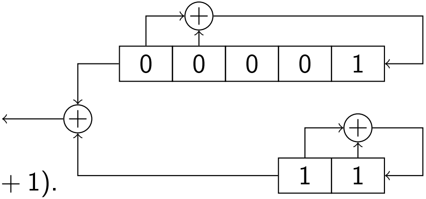

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All three sequences of period 21 would have turned out to be the same!

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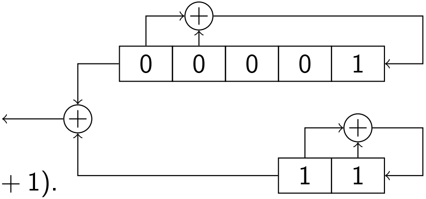

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We also have sequences of periods 7,3, and 1, reaching 2^5.

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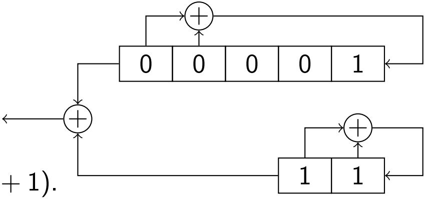

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Do the following to analyze LFSRs:

1. Factor the characteristic polynomial $P(x) = \prod f_i^{e_i}(x)$, for $f_i(x)$ irreducible, $f_i \neq f_j$, and $e_i > 0$.
2. Compute orders of $f_i^{e_i}(x)$.
3. Combine periods, taking care of offsets to get all periods. No cancellations because the f_i are co-prime.

Step 2 is different from what you did on sheet 2. Revisit LFSR (f).

Correct hypotheses

The following holds for LFSRs with co-prime characteristic polynomials.

- Adding LFSRs of max periods p and r gives period $\mathrm{lcm}(p, r)$.
- If the first LFSR has periods $p = 2^m - 1$ and 1 and the second LFSR has periods $r = 2^n - 1$ and 1, then
  - their sum has $\gcd(p, r)$ sequences of period $\mathrm{lcm}(p, r)$ (resulting from the $\gcd(p, r)$ different offsets)
  - and sequences of period p, r, and 1, from initializing one or both in the all-zero state.
  - These sum up to $\gcd(p, r) \cdot \mathrm{lcm}(p, r) + p + r + 1 = p \cdot r + p + r + 1$ $= (p + 1)(r + 1) = 2^m \cdot 2^n$, thus accounting for all 2^{m+n} states.
- If one or both do not have maximal periods we expect
  - $\gcd(p, r)$ sequences of period $\mathrm{lcm}(p, r)$
  - sequences of period p, r, and 1,
  - sequences from combinations of the other parts.