Sums of LFSRs

Tanja Lange

Eindhoven University of Technology

2WF80: Introduction to Cryptology

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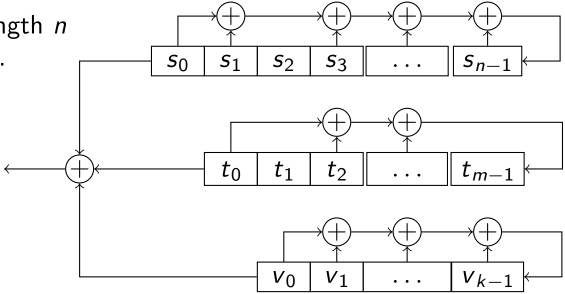
LFSR with state of length n has period at most 2^n.



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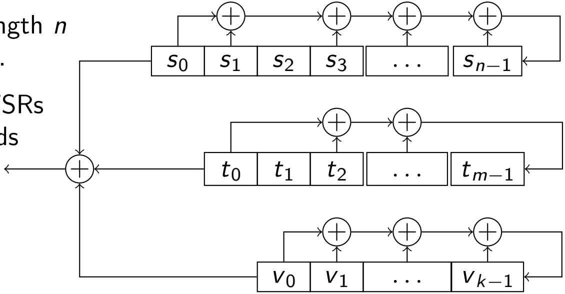

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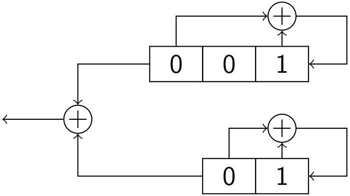

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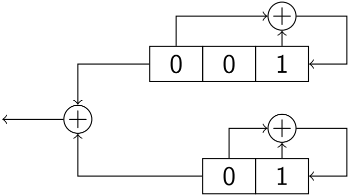

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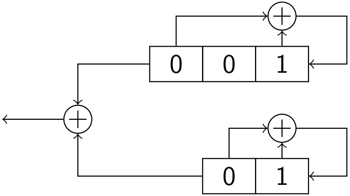

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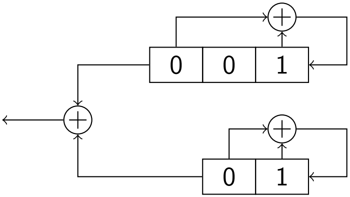

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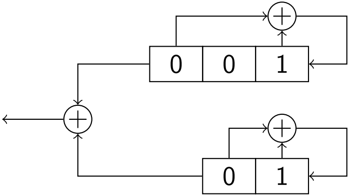
```



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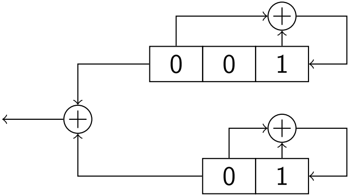
  0 1 0 1 0 1 1 1 1 1 0 0 0 0
```



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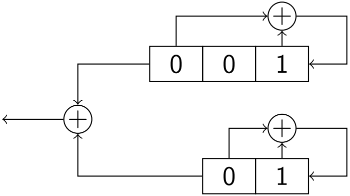


```
  0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
-------------------------------------------
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```

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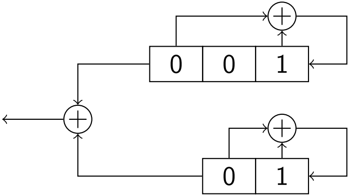

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```
  0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1
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-------------------------------------------
  0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0 0 1 1 0
```

These LFSRs of periods 3 and 7 combine to period $3 \cdot 7 = 21$

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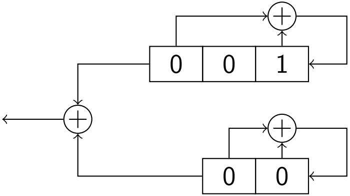

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```
  0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
--------------------------------------------
  0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0 0 1 1 0
```

These LFSRs of periods 3 and 7 combine to period $3 \cdot 7 = 21$, 7

```
  0 0 1 1 1 0 1
+ 0 0 0 0 0 0 0
----------------
  0 0 1 1 1 0 1
```

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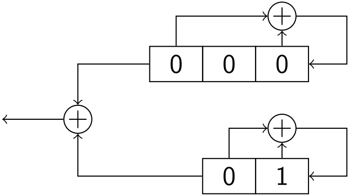

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```
  0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
--------------------------------------------
  0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0 0 1 1 0
```

These LFSRs of periods 3 and 7 combine to period $3 \cdot 7 = 21$, 7, 3

```
  0 0 1 1 1 0 1          0 0 0
+ 0 0 0 0 0 0 0        + 0 1 1
---------------        -------
  0 0 1 1 1 0 1          0 1 1
```

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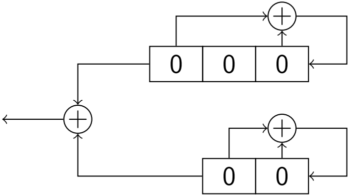


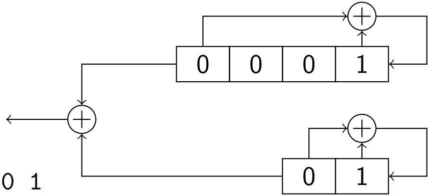

```
  0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
-------------------------------------------
  0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0 0 1 1 0
```

These LFSRs of periods 3 and 7 combine to period $3 \cdot 7 = 21$, 7, 3, 1.

```
  0 0 1 1 1 0 1            0 0 0          0
+ 0 0 0 0 0 0 0          + 0 1 1        + 0
---------------          -------        ---
  0 0 1 1 1 0 1            0 1 1          0
```

Another example

These LFSRs produce
$\overline{000111101011001}$ and $\overline{011}$
of periods 15 and 3.



Their sum gives

	0	0	0	1	1	1	1	0	1	0	1	1	0	0	1
+	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
	0	1	1	1	0	0	1	1	0	0	0	0	0	1	0

of period 15.

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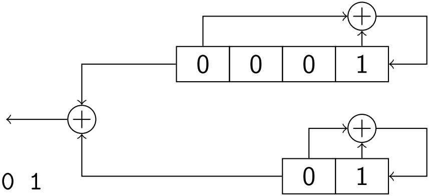

Their sum gives

	0	0	0	1	1	1	1	0	1	0	1	1	0	0	1
+	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
	0	1	1	1	0	0	1	1	0	0	0	0	0	1	0

of period 15.

Initializing one or both LFSRs with all-zero state gives 15, 3, 1 – but we expect $2^6 = 64$ states.

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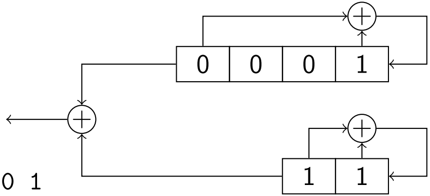

Their sum gives

```
  0 0 0 1 1 1 1 0 1 0 1 1 0 0 1
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---------------------------------
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```

of period 15.

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```
  0 0 0 1 1 1 1 0 1 0 1 1 0 0 1
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---------------------------------
  1 1 0 0 0 1 0 1 1 1 0 1 1 1 1
```

Starting at different offsets gives periods 15

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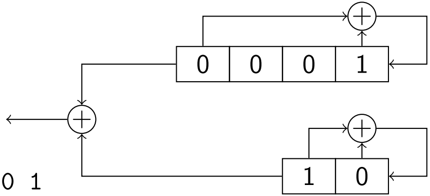

Their sum gives

```
  0 0 0 1 1 1 1 0 1 0 1 1 0 0 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
--------------------------------
  0 1 1 1 0 0 1 1 0 0 0 0 0 1 0
```

of period 15.

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```
  0 0 0 1 1 1 1 0 1 0 1 1 0 0 1
+ 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1
--------------------------------
  1 0 1 0 1 0 0 0 0 1 1 0 1 0 0
```

Starting at different offsets gives periods 15 and 15.

For a total of periods
15, 15, 15, 15, 3, 1,
summing up to 64.

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  - and sequences of period p, r, and 1, from initializing one or both in the all-zero state.
  - These sum up to $\gcd(p, r) \cdot \operatorname{lcm}(p, r) + p + r + 1 = p \cdot r + p + r + 1$ $= (p + 1)(r + 1) = 2^m \cdot 2^n$, thus accounting for all 2^{m+n} states.

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  - These sum up to $\gcd(p, r) \cdot \text{lcm}(p, r) + p + r + 1 = p \cdot r + p + r + 1$
    $= (p + 1)(r + 1) = 2^m \cdot 2^n$,
    thus accounting for all 2^{m+n} states.
- If one or both do not have maximal periods we expect
  - $\gcd(p, r)$ sequences of period $\text{lcm}(p, r)$
  - sequences of period p, r, and 1,
  - sequences from combinations of the other parts.

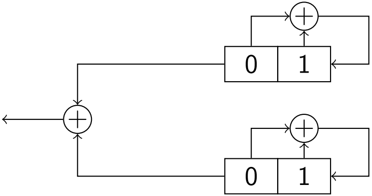
A third example

These LFSRs produce
$\overline{011}$ and $\overline{011}$
of periods 3 and 3.

Their sum gives

```
  0 1 1
+ 0 1 1
-------
  0 0 0
```

of period 1.



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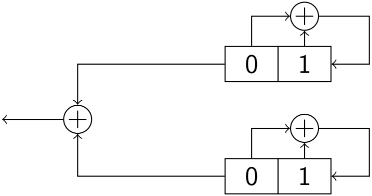

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of period 1. The same will happen whenever the starting states are equal.

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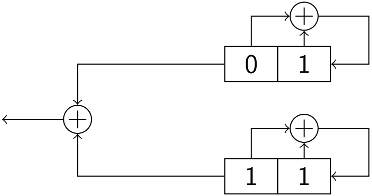

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```
  0 1 1
+ 0 1 1
-------
  0 0 0
```

of period 1. The same will happen whenever the starting states are equal.

Shifting one starting state gives

```
  0 1 1
+ 1 1 0
-------
  1 0 1
```

of period 3.

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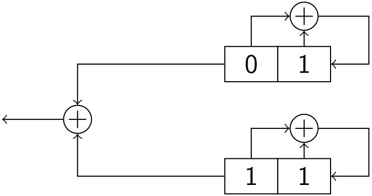

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```

of period 1. The same will happen whenever the starting states are equal.

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```
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```

of period 3. This is the same sequence as just one of them.

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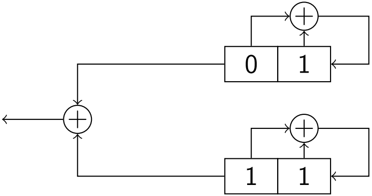

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```
  0 1 1
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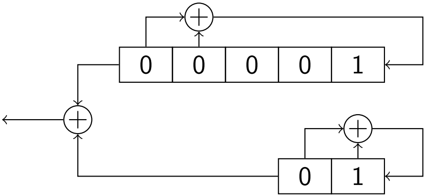
Not useful to combine identical LFSRs.

A fourth example



These LFSRs produce $\overline{000010001100101011111}$ and $\overline{011}$ of periods 21 and 3.

Their sum gives

```
  0 0 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 1 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
-------------------------------------------
  0 1 1 0 0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0
```

of period 21.

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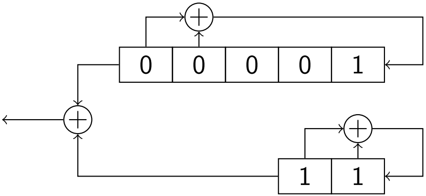

Their sum gives

```
  0 0 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 1 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
--------------------------------------------
  0 1 1 0 0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0
```

of period 21.

```
  0 0 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 1 1
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--------------------------------------------
  1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1
```

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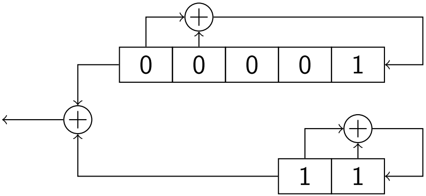

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-------------------------------------------
  0 1 1 0 0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0
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of period 21.

```
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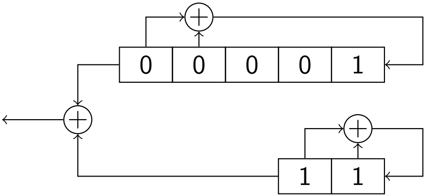

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of period 21.

```
  0 0 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 1 1
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  1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1
```

of period 7?

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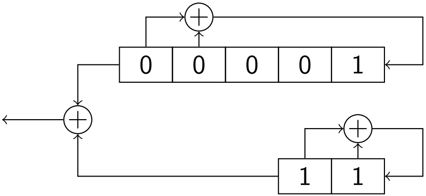

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```
  0 0 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 1 1
+ 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
-------------------------------------------
  0 1 1 0 0 1 0 1 0 1 1 1 1 1 0 0 0 0 1 0 0
```

of period 21.

```
  0 0 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 1 1
+ 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0 1 1 0
-------------------------------------------
  1 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1
```

of period 7?

Our hypotheses would have predicted: 21, 21, 21, 21, 3, 1 and some more for the $2^5 - 21 - 1 = 10$ missing states in the first. But we do not get the fourth 21.