Workshop on Code-based Cryptography

Eindhoven, May 11 & 12, 2011

For details see

http://www.win.tue.nl/cccc/cbc/

Breaking ECC2K-130

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Ruben Niederhagen, Christof Paar, Francesco Regazzoni,
Peter Schwabe, Leif Uhsadel, Anthony Van Herrewege,
Bo-Yin Yang,
and several individuals and institutions donating computer time

2010.04.15

The Certicom challenges

1997: Certicom announces several ECDLP prizes:

> *The Challenge is to compute the ECC private keys from the given list of ECC public keys and associated system parameters. This is the type of problem facing an adversary who wishes to completely defeat an elliptic curve cryptosystem.*

Objectives stated by Certicom:

- Increase community's understanding of ECDLP difficulty.
- Confirm theoretical comparisons of ECC and RSA.
- Help users select suitable key sizes.
- Compare ECDLP difficulty for $\mathbf{F}_{2^m}$ and $\mathbf{F}_p$.
- Compare $\mathbf{F}_{2^m}$ ECDLP difficulty for random and Koblitz.
- Stimulate research in algorithmic number theory.

The Certicom challenges, level 0: exercises

Bits	Name	"Estimated number of machine days"	Prize
79	ECCp-79	146	book
79	ECC2-79	352	book
89	ECCp-89	4360	book
89	ECC2-89	11278	book
97	ECC2K-95	8637	$5000
97	ECCp-97	71982	$5000
97	ECC2-97	180448	$5000

Certicom believes that it is feasible that the 79-bit exercises could be solved in a matter of hours, the 89-bit exercises could be solved in a matter of days, and the 97-bit exercises in a matter of weeks using a network of 3000 computers.

The Certicom challenges, level 1

Bits	Name	“Estimated number of machine days”	Prize
109	ECC2K-108	1300000	$10000
109	ECCp-109	9000000	$10000
109	ECC2-109	21000000	$10000
131	ECC2K-130	2700000000	$20000
131	ECCp-131	23000000000	$20000
131	ECC2-131	66000000000	$20000

The 109-bit Level I challenges are feasible using a very large network of computers. The 131-bit Level I challenges are expected to be infeasible against realistic software and hardware attacks, unless of course, a new algorithm for the ECDLP is discovered.

The Certicom challenges, level 2

Bits	Name	"Estimated number of machine days"	Prize
163	ECC2K-163	320000000000000	$30000
163	ECCp-163	2300000000000000	$30000
163	ECC2-163	6200000000000000	$30000
191	ECCp-191	48000000000000000000	$40000
191	ECC2-191	100000000000000000000	$40000
239	ECC2K-238	920000000000000000000000000	$50000
239	ECCp-239	14000000000000000000000000000	$50000
239	ECC2-238	21000000000000000000000000000	$50000
359	ECCp-359	$\approx \infty$	$100000

The Level II challenges are infeasible given today's computer technology and knowledge.

Broken challenges

1997: Baisley and Harley break ECCp-79.
1997: Harley et al. break ECC2-79.
1998: Harley et al. break ECCp-89.
1998: Harley et al. break ECC2-89.
1998: Harley et al. (1288 computers) break ECCp-97.
1998: Harley et al. (200 computers) break ECC2K-95.
1999: Harley et al. (740 computers) break ECC2-97.
2000: Harley et al. (9500 computers) break ECC2K-108.
2002: Monico et al. (10000 computers) break ECCp-109.
2004: Monico et al. (2600 computers) break ECC2-109.

Updated 2003 document `cert_ecc_challenge.pdf` still said "109-bit Level I challenges are feasible using a very large network . . . 131-bit Level I challenges are expected to be infeasible" etc.

The Certicom challenges ECC2-X

VAM1 research retreat in Lausanne on SHARCS topics.

Decision to analyze the Certicom challenges ECC2K-130, ECC2-131, ECC2K-163, ECC2-163.
Can we break ECC2K-130?
"Infeasible" sounds tempting.

Direct effects:

- Certicom backpedals. Withdraws "infeasible" statement. Instead says that ECC2K-130 "may be within reach."

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Direct effects:

- Certicom backpedals. Withdraws "infeasible" statement. Instead says that ECC2K-130 "may be within reach."
- ECRYPT has several new research papers, starting with paper at SHARCS "The Certicom challenges ECC2-X."

The target: ECC2K-130

The Koblitz curve $y^2 + xy = x^3 + 1$ over
$\mathbf{F}_{2^{131}} = \mathbf{F}_2[z]/(z^{131} + z^{13} + z^2 + z + 1)$
has 4ℓ points, where ℓ is the prime
$680564733841876926932320129493409985129 \approx 2^{129}$.

Certicom generated two random points on the curve
and multiplied them by 4, obtaining the following points P, Q:

```
x(P) = 05 1C99BFA6 F18DE467 C80C23B9 8C7994AA
y(P) = 04 2EA2D112 ECEC71FC F7E000D7 EFC978BD
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y(Q) = 04 A38D1182 9D32D347 BD0C0F58 4D546E9A
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The challenge:
Find an integer $k \in \{0, 1, \ldots, \ell - 1\}$ such that $[k]P = Q$.
Worthy target:

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Worthy target: \$ 20000 (but only CAD)

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Worthy target: 128-bit curves have been proposed for real
(RFID, TinyTate).

Arithmetic on ECC2K-130

Elements of the Koblitz curve: a special point P_∞, and each $(x_1, y_1) \in \mathbf{F}_{2^{131}} \times \mathbf{F}_{2^{131}}$ satisfying $y_1^2 + x_1 y_1 = x_1^3 + 1$.

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How to add P_1, P_2:

- $P_1 + P_\infty = P_\infty + P_1 = P_1$; $(x_1, y_1) + (x_1, y_1 + x_1) = P_\infty$.
- If $x_1 \neq 0$ the double $[2](x_1, y_1) = (x_3, y_3)$ is given by

$$x_3 = \lambda^2 + \lambda,\ \ y_3 = \lambda(x_1 + x_3) + y_1 + x_3,\ \ \text{where } \lambda = x_1 + \frac{y_1}{x_1}.$$

- If $x_1 \neq x_2$ the sum $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ is given by

$$x_3 = \lambda^2 + \lambda + x_1 + x_2,\ \ y_3 = \lambda(x_1 + x_3) + y_1 + x_3,\ \ \text{where } \lambda = \frac{y_1 + y_2}{x_1 + x_2}.$$

  Cost: 1**I** (inversion), 2**M** (multiplications), 1**S** (squaring).
- For an overview of how to perform these operations in other coordinate systems see the EFD:

  `http://hyperelliptic.org/EFD/`

  and upcoming talk.

Koblitz curves – the Frobenius endomorphism

- In 1991 Koblitz pointed out that scalar multiplications $[m]P$ can be computed faster on curves

$$E_a : y^2 + xy = x^3 + ax^2 + 1,$$

where a is restricted to $\{0, 1\}$.

- The main observation is that if $(x_1, y_1) \in E_a(\mathbf{F}_{2^n})$ then also the point $\sigma(P) = (x_1^2, y_1^2)$ is in $E_a(\mathbf{F}_{2^n})$ and these points are related by

$$\sigma^2(P) + [\mu]\sigma(P) + [2]P = P_\infty,$$

where $\mu = 1$ for $a = 0$ and $\mu = -1$ for $a = 1$. The map σ extends the Frobenius automorphism of $\mathbf{F}_{2^n}$ to $E_a(\mathbf{F}_{2^n})$ and is thus called the Frobenius endomorphism of E_a.

Koblitz curves – usage of σ

- Koblitz, Meyer–Staffelbach, and Solinas showed that in the computation of $[m]P$ the double-and-add method can be replaced by a σ-and-add method. Instead of needing $\log_2 m$ doublings the Frobenius-based method needs $\log_2 m$ applications of σ.
  - This means that instead of $1\mathbf{I} + 2\mathbf{M} + 1\mathbf{S}$ per bit of m only $2\mathbf{S}$ are needed per bit of m.
- The cost per addition does not change for these curves.
  - A NAF version reduces the number of additions to $\log_2 m/3$ on average without needing any precomputations.
  - Analogues of (binary) windowing methods exist in Frobenius variants.

The most important ECDL algorithms

No known index-calculus attack applies to ECC2K-130.
But can still use generic attacks that work in any group:

- The Pohlig–Hellman attack reduces the hardness of the ECDLP to the hardness of the ECDLP in the largest subgroup of prime order: in this case order ℓ.
- The Baby-Step Giant-Step attack finds the logarithm in $\sqrt{\ell}$ steps and $\sqrt{\ell}$ storage by comparing $Q - [jt]P$ (the giant steps) to a sorted list of all $[i]P$ (the baby steps), where $0 \leq i, j \leq \lceil \sqrt{\ell} \rceil$ and $t = \lceil \sqrt{\ell} \rceil$.
- Pollard's rho and kangaroo methods also use $O(\sqrt{\ell})$ steps but require constant memory—much less expensive! The kangaroo method would be faster if the logarithm were known to lie in a short interval; for us rho is best.

Lots of slides on my homepage from recent course on ECDLP.

Pollard's rho method

Make a pseudo-random walk in $\langle P \rangle$, where the next step depends on current point: $P_{i+1} = f(P_i)$.

Birthday paradox: Randomly choosing from ℓ elements picks one element twice after about $\sqrt{\pi\ell/2}$ draws.

The walk has now entered a cycle.
Cycle-finding algorithm (e.g., Floyd) quickly detects this.

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Assume that for each point we know $a_i, b_i \in \mathbf{Z}/\ell\mathbf{Z}$ so that $P_i = [a_i]P + [b_i]Q$. Then $P_i = P_j$ means that

$$[a_i]P + [b_i]Q = [a_j]P + [b_j]Q \quad \text{so} \quad [b_i - b_j]Q = [a_j - a_i]P.$$

If $b_i \neq b_j$ the ECDLP is solved: $k = (a_j - a_i)/(b_i - b_j)$.

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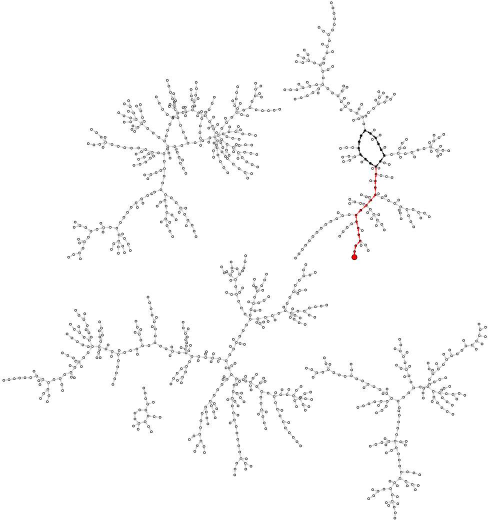
e.g. "Adding walk": Start with $P_0 = P$ and put $f(P_i) = P_i + [c_r]P + [d_r]Q$ where $r = h(P_i)$.

A rho within a random walk on 1024 elements



Method is called rho method because of the shape.

Parallel collision search

Running Pollard's rho method on N computers gives speedup of $\approx \sqrt{N}$ from increased likelihood of finding collision.

Want better way to spread computation across clients.
Want to find collisions between walks on different machines, without frequent synchronization!

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Want better way to spread computation across clients. Want to find collisions between walks on different machines, without frequent synchronization!

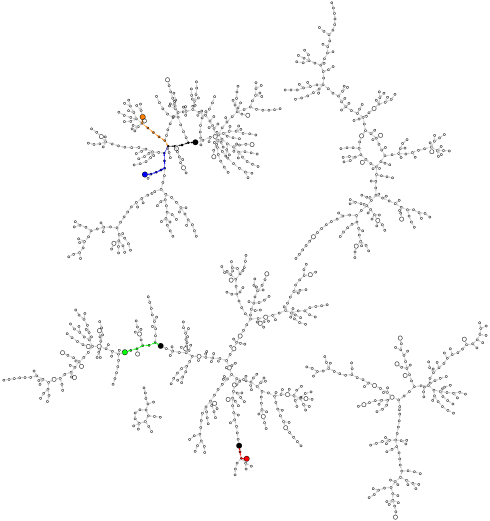
Perform walks with different starting points but same update function on all computers. If same point is found on two different computers also the following steps will be the same.

Terminate each walk once it hits a distinguished point. Attacker chooses definition of distinguished points; can be more or less frequent. Do not wait for cycle.

Collect all distinguished points in central database.

Expect collision within $O(\sqrt{\ell}/N)$ iterations. Speedup $\approx N$.

Short walks ending in distinguished points



Blue and orange paths found the same distinguished point!

Equivalence classes

P and $-P$ have same x-coordinate. Search for x-coordinate collision. Search space for collisions is only $\ell/2$; this gives factor $\sqrt{2}$ speedup ... provided that $f(P_i) = f(-P_i)$.

Solution: $f(P_i) = |P_i| + [c_r]P + [d_r]Q$ where $r = h(|P_i|)$. Define $|P_i|$ as, e.g., lexicographic minimum of $P_i, -P_i$.

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Problem: this walk can run into fruitless cycles!
If there are S different steps $[c_r]P + [d_r]Q$ then with probability $1/(2S)$ the following happens for some step:

$$\begin{aligned} P_{i+2} &= P_{i+1} + [c_r]P + [d_r]Q \\ &= -(P_i + [c_r]P + [d_r]Q) + [c_r]P + [d_r]Q = -P_i, \end{aligned}$$

i.e. $|P_i| = |P_{i+2}|$. Get $|P_{i+3}| = |P_{i+1}|$, $|P_{i+4}| = |P_i|$, etc.
Can detect and fix, but requires attention.
See PKC 2011 paper for how to do this over $\mathbf{F}_p$.

Equivalence classes for Koblitz curves

More savings: P and $\sigma^i(P)$ have $x(\sigma^j(P)) = x(P)^{2^j}$.

Reduce number of iterations by another factor $\sqrt{n}$ by considering equivalence classes under Frobenius and $\pm$.

Need to ensure that the iteration function satisfies $f(P_i) = f(\pm\sigma^j(P_i))$ for any j.

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Need to ensure that the iteration function satisfies $f(P_i) = f(\pm\sigma^j(P_i))$ for any j.

Could again define adding walk starting from $|P_i|$.
Redefine $|P_i|$ as canonical representative of class containing P_i: e.g., lexicographic minimum of P_i, $-P_i$, $\sigma(P_i)$, etc.

Iterations now involve many squarings,
but squarings are not so expensive in characteristic 2.

Iteration functions for Koblitz curves

Harley and Gallant-Lambert-Vanstone observe that in normal basis, $x(P)$ and $x(P)^{2^j}$ have same Hamming weight $\mathrm{HW}(x(P))$ and suggest to use

$$P_{i+1} = P_i + \sigma^j(P_i),$$

as iteration function. Choice of j depends on $\mathrm{HW}(x(P))$. This ensures that the walk is well defined on classes since

$$\begin{aligned} f(\pm\sigma^m(P_i)) &= \pm\sigma^m(P_i) + \sigma^j(\pm\sigma^m(P_i)) \\ &= \pm(\sigma^m(P_i) + \sigma^m(\sigma^j(P_i))) = \pm\sigma^m(P_{i+1}). \end{aligned}$$

- GLV suggest using $j = \mathrm{hash}(\mathrm{HW}(x(P)))$, where the hash function maps to $[1, n]$.
- Harley uses a smaller set of exponents; for his attack on ECC2K-108 he takes $j \in \{1, 2, 4, 5, 6, 7, 8\}$; computed as $(\mathrm{HW}(x(P)) \bmod 7) + 2$ and replacing 3 by 1.

Our choice of iteration function I

Restricting size of j matters – squarings are cheap but

- in bitslicing need to compute all powers (no branches allowed);
- code size matters (in particular for Cell CPU);
- logic costs area for FPGA;
- having a large set doesn't actually gain much randomness (see analysis coming up).

Having few coefficients makes it possible to exclude short fruitless cycles. To do so, compute the shortest vector in the lattice $\left\{v : \prod_j (1+\sigma^j)^{v_j} = 1\right\}$. Usually the shortest vector has negative coefficients (which cannot happen with the iteration); shortest vector with positive coefficients is somewhat longer. For implementation it is better to have a continuous interval of exponents, so shift the interval if shortest vector is short.

Our choice of iteration function II

Our iteration function:

$$P_{i+1} = P_i + \sigma^j(P_i),$$

where $j = (\mathrm{HW}(x(P))/2 \bmod 8) + 3$, so $j \in \{3, 4, 5, 6, 7, 8, 9, 10\}$. Shortest combination of these powers is long. Note that $\mathrm{HW}(x(P))$ is always even.

Iteration consists of

- computing the Hamming weight $\mathrm{HW}(x(P))$ of the normal-basis representation of $x(P)$;
- checking for distinguished points (is $\mathrm{HW}(x(P)) \leq 34$?);
- computing j and $P + \sigma^j(P)$.

This choice of iteration function avoids fruitless cycles – Koblitz curves save factor of $\sqrt{2n}$ *and* avoid problems dealing with cycles.

Analysis of our choice of iteration function

For a perfectly random walk $\approx \sqrt{\pi\ell/2}$ iterations are expected on average. Have $\ell \approx 2^{131}/4$ for ECC2K-130.

A perfectly random walk on classes under $\pm$ and Frobenius would reduce number of iterations by $\sqrt{2 \cdot 131}$.

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Loss of randomness from having only 8 choices of j.
Further loss from non-randomness of Hamming weights: Hamming weights around 66 are much more likely than at the edges; effect still noticeable after reduction to 8 choices.

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Our heuristic analysis says that the total loss is 6.9993%.
(Very new "anti-collision" analysis: actually above 7%.)
This loss is justified by the very fast iteration function.

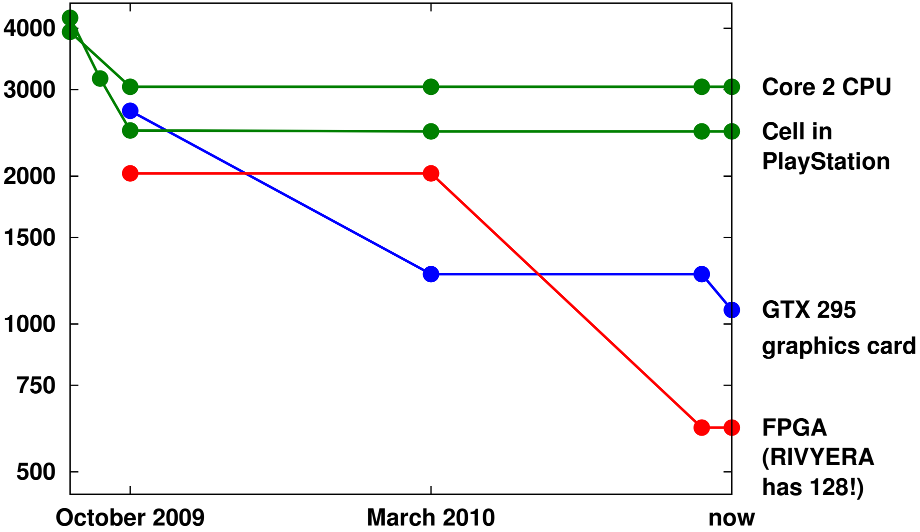
Average number of iterations for our attack against ECC2K-130: $\sqrt{\pi \ell / (2 \cdot 2 \cdot 131)} \cdot 1.069993 \approx 2^{60.9}$.

Some highlights

- Detailed analysis of randomness of iteration function.
- Could increase randomness of the walk but then iteration function gets slower. Optimized:

  $$\text{time per iteration} \times \#\ \text{iterations}$$
- Do not remember multiset of j's; instead recompute this from seed when collision is found (cheaper, less storage).
- Comparative study of normal basis and polynomial basis representation; new: optimal polynomial bases.
- For Cell processor (chip in PlayStation 3) fierce battle between bitsliced and non-bitsliced implementation. Result: much faster implementation! (Bitsliced won.)
- Assembly language for GPUs and qhasm version. Get control over powerful beast.
- FPGA implementation of Shokrollahi multiplier: big speed-up, useful also for constructive ECC.

Faster implementations



Number of cards or chips needed for $68 \cdot 10^9$ iterations/second.

Running the attack

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To prove it we're running the attack.

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Eight central servers receive points, pre-sort the points into 8192 RAM buffers, flush the buffers to 8192 disk files.

Periodically read each file into RAM, sort, find collisions.
Also double-check random samples for validity.

Several sites contribute points, including several clusters. E.g. test runs on first generation of PRACE clusters

http://www.prace-project.eu

Each packet is encrypted, authenticated, verified, decrypted using http://nacl.cace-project.eu; costs 16 bytes.
Total block cost: 1090-byte IP packet plus 66-byte ack.

Get more details, and watch our progress!

http://ecc-challenge.info
https://twitter.com/ECCchallenge

Papers and preprints:

- "The Certicom challenges ECC2-X" (SHARCS 2009)
- "ECC2K-130 on Cell CPUs" (AFRICACRYPT 2010)
- "Type-II optimal polynomial bases" (WAIFI 2010)
- "Breaking elliptic curve cryptosystems using reconfigurable hardware" (FPL 2010)
- "ECC2K-130 on NVIDIA GPUs" (INDOCRYPT 2010)
- "Usable assembly language for GPUs"
- "Anti-collisions in Pollard's rho method"
- The whole attack in progress: "Breaking ECC2K-130"