ECDLP course

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Main goal of this course: We are the attackers. We want to break ECC.

Enemy: ECC users.
Helpful for us as attackers: Understand the enemy!

What are ECC users doing? Why are they doing this? What will they do tomorrow?

(This perspective also helps if you want to use ECC.)
Can break ECC in many ways. But most of these ways can be easily stopped by ECC users.

Most important strategy for attacking ECC:
Break ECDLP by parallel Pollard rho with all available refinements.

ECC users cannot stop rho, although they can make it slower by increasing ECC key size.
Why ECC?

January 2010 news: An academic team announces successful RSA-768 factorization. Used $\approx 2$ years of computation on $\approx 1000$ CPU cores. “Factoring a 1024-bit RSA modulus would be about a thousand times harder.”
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Many users of 1024-bit RSA: EFF SSL observatory study http://eff.org/observatory shows that more than 50% of all certificates are $\leq 1024$-bit RSA.
1000 cores in perspective:

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Dan has an account on the TACC Ranger supercomputer, which has 62976 cores.
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Dan has an account on the TACC Ranger supercomputer, which has 62976 cores.
The Conficker/Downadup criminal-controlled botnet has $\approx 10\,000\,000$ cores.
2003 Shamir et al.: An attacker building ASICs for 10 million USD can break RSA-1024 in a year.

2003 RSA company: Move to 2048 bits “over the remainder of this decade.”
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2007 NIST: Same.

These recommendations don’t even take into account batch algorithms that save time in breaking many keys together.
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Still below modern standards!
Attacks: \( \approx 2^{112} \) calculations.

3072-bit key: 1536-bit primes;
\( \approx 2^{1526} \) possible primes.
Attacks: \( \approx 2^{128} \) calculations.
Attacks use “index calculus” = “combining congruences.” Long history, including many major improvements: 1975, CFRAC; 1977, linear sieve (LS); 1982, quadratic sieve (QS); 1990, number-field sieve (NFS). Also many smaller improvements.

Costs of these algorithms for breaking RSA-1024, RSA-2048:
\[2^{120}, 2^{170}, \text{CFRAC};\]
\[2^{110}, 2^{160}, \text{LS};\]
\[2^{100}, 2^{150}, \text{QS};\]
\[2^{80}, 2^{112}, \text{NFS}.\]
1977: RSA is introduced.

1985: Miller proposes ECC. Explains several obstacles to congruence-combination attacks on elliptic curves.

Subsequent ECC history: Improved algorithms, but still take exponential time.

Subsequent RSA history: Continued security losses from improved algorithms for combining congruences. Major loss in 1990 (NFS); many smaller losses since then.
256-bit ECC keys match security of 3072-bit RSA keys. When properly implemented, 256-bit ECC is much faster than 3072-bit RSA for almost all real-world applications.

ANSI, IEEE, NIST issued ECC standards ten years ago. US government “Suite B” now prohibits RSA, requires ECC. Electronic ID or travel documents in many European countries based on ECC. Some internet protocols (DNSCurve, CurveCP) use ECC.
The clock

This is the curve $x^2 + y^2 = 1$.

Warning:
This is not an elliptic curve.
“Elliptic curve” $\neq$ “ellipse.”
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\[(0, 1) = \text{“12:00”}.\]

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$(\sqrt{3}/4, 1/2) =$
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\((-1, 0) = \text{“9:00”}\).
\((\sqrt{3/4}, 1/2) = \text{“2:00”}\).
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\((\sqrt{3}/4, 1/2) = \text{“2:00”}\).
\((1/2, -\sqrt{3}/4) = \text{“5:00”}\).
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\((-1/2, -\sqrt{3}/4) = \text{“7:00”}\).
Examples of points on this curve:

(0, 1) = “12:00”.
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(−1, 0) = “9:00”.

(\sqrt{3/4}, 1/2) = “2:00”.
(1/2, −\sqrt{3/4}) = “5:00”.
(−1/2, −\sqrt{3/4}) = “7:00”.
(\sqrt{1/2}, \sqrt{1/2}) = “1:30”.

(3/5, 4/5). (−3/5, 4/5).
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$(0, 1) = "12:00".$

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$(3/5, 4/5). (-3/5, 4/5).$

$(3/5, -4/5). (-3/5, -4/5).$

$(4/5, 3/5). (-4/5, 3/5).$

$(4/5, -3/5). (-4/5, -3/5).$

Many more.
Addition on the clock:

neutral = (0, 1)

$P_1 = (x_1, y_1)$

$P_2 = (x_2, y_2)$

$P_3 = (x_3, y_3)$

$x^2 + y^2 = 1$, parametrized by $x = \sin \alpha$, $y = \cos \alpha$. 
Addition on the clock:

\[ \text{neutral} = (0, 1) \]

\[ P_1 = (x_1, y_1) \]

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\[ P_3 = (x_3, y_3) \]

\[ x^2 + y^2 = 1, \text{ parametrized by} \]

\[ x = \sin \alpha, \quad y = \cos \alpha. \text{ Recall} \]

\[ (\sin(\alpha_1 + \alpha_2), \cos(\alpha_1 + \alpha_2)) = \]
Addition on the clock:

\[ x^2 + y^2 = 1, \] parametrized by \[ x = \sin \alpha, \quad y = \cos \alpha. \] Recall
\[ (\sin(\alpha_1 + \alpha_2), \cos(\alpha_1 + \alpha_2)) = (\sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \sin \alpha_2, \]
Addition on the clock:

\[ x^2 + y^2 = 1, \] parametrized by
\[ x = \sin \alpha, \quad y = \cos \alpha. \]
Recall
\[ (\sin(\alpha_1 + \alpha_2), \cos(\alpha_1 + \alpha_2)) = \]
\[ (\sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \sin \alpha_2, \]
\[ \cos \alpha_1 \cos \alpha_2 - \sin \alpha_1 \sin \alpha_2). \]
Adding two points corresponds to adding the angles $\alpha_1$ and $\alpha_2$. Angles modulo $360^\circ$ are a group, so points on clock are a group.

Neutral element: angle $\alpha = 0$; point $(0, 1)$; “12:00”.

The point with $\alpha = 180^\circ$ has order 2 and equals 6:00. 3:00 and 9:00 have order 4.

Inverse of point with $\alpha$ is point with $-\alpha$ since $\alpha + (-\alpha) = 0$.

There are many more points where angle $\alpha$ is not “nice.”
Clock addition without sin, cos:

Use Cartesian coordinates for addition. Addition formula for the clock $x^2 + y^2 = 1$:

\[
\text{sum } (x_1, y_1) + (x_2, y_2) = (x_3, y_3) = (x_1 y_2 + y_1 x_2, y_1 y_2 - x_1 x_2).
\]

Note $(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

$kP = P + P + \cdots + P$ for $k \geq 0$. 

$k$ copies
Examples of clock addition:

“2:00” + “5:00”
= \((\sqrt{3}/4, 1/2) + (1/2, -\sqrt{3}/4)\)
= \((-1/2, -\sqrt{3}/4) = “7:00”.

“5:00” + “9:00”
= \((1/2, -\sqrt{3}/4) + (-1, 0)\)
= \((\sqrt{3}/4, 1/2) = “2:00”.

\[2 \begin{pmatrix} 3 & 4 \\ \frac{5}{5} & \frac{5}{5} \end{pmatrix} = \begin{pmatrix} 24 & 7 \\ \frac{25}{25} & \frac{25}{25} \end{pmatrix}.\]
Examples of clock addition:

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\[
2 \begin{pmatrix}
3 \\
5', 5
\end{pmatrix} = \begin{pmatrix}
24 \\
25', 25
\end{pmatrix}.
\]

\[
3 \begin{pmatrix}
3 \\
5', 5
\end{pmatrix} = \begin{pmatrix}
117 \\
125', 125
\end{pmatrix}.
\]
Examples of clock addition:

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\[
2 \begin{pmatrix} 3 \\ 5' \\ 5 \end{pmatrix} = \begin{pmatrix} 24 \\ 25' \\ 25 \end{pmatrix}.
\]

\[
3 \begin{pmatrix} 3 \\ 5' \\ 5 \end{pmatrix} = \begin{pmatrix} 117 \\ 125' \\ 125 \end{pmatrix}.
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\[
4 \begin{pmatrix} 3 \\ 5' \\ 5 \end{pmatrix} = \begin{pmatrix} 336 \\ 625' \\ 625 \end{pmatrix}.
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Examples of clock addition:

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\[ 4 \begin{pmatrix} 3 & 4 \\ 5 & 5 \end{pmatrix} = \begin{pmatrix} 336 & -527 \\ 625 & 625 \end{pmatrix} . \]

\((x_1, y_1) + (0, 1) = \)
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Examples of clock addition:

“2:00” + “5:00”
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“5:00” + “9:00”
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\overline{5}, & \overline{5}
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\((x_1, y_1) + (0, 1) = (x_1, y_1)\).

\((x_1, y_1) + (-x_1, y_1) = (0, 1)\).
Clocks over finite fields

Clock($\mathbf{F}_7$) =
\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2 = 1\}.
Here $\mathbf{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$
= \{0, 1, 2, 3, -3, -2, -1\}
with $+,-,\times$ modulo 7.
Larger example: \( \text{Clock}(F_{1000003}) \).

Examples of clock addition:

\[
2(1000, 2) = (4000, 7).
\]

\[
4(1000, 2) = (56000, 97).
\]

\[
8(1000, 2) = (863970, 18817).
\]

\[
16(1000, 2) = (549438, 156853).
\]

\[
17(1000, 2) = (951405, 877356).
\]

With 30 clock additions we computed

\[
n(1000, 2) = (947472, 736284)
\]

for some 6-digit \( n \).

Can you figure out \( n \)?
Clock cryptography

Standardize a large prime $p$ and some $(X, Y) \in \text{Clock}(\mathbb{F}_p)$.

Alice chooses big secret $a$.
Computes her public key $a(X, Y)$.

Bob chooses big secret $b$.
Computes his public key $b(X, Y)$.

Alice computes $a(b(X, Y))$.
Bob computes $b(a(X, Y))$.
I.e., both obtain $(ab)(X, Y)$.
They use this shared value to encrypt with AES-GCM etc.
Alice’s secret key $a$

\[
downarrow
\]

Alice’s public key $a(X, Y)$

\{Alice, Bob\}'s shared secret $ab(X, Y)$

\{Bob, Alice\}'s shared secret $ba(X, Y)$

Bob’s secret key $b$

\[
downarrow
\]

Bob’s public key $b(X, Y)$
Alice’s secret key $a$

\[ \downarrow \]

Alice’s public key $a(X, Y)$

\[ \downarrow \]

\{Alice, Bob\}’s shared secret $ab(X, Y)$

\{Bob, Alice\}’s shared secret $ba(X, Y)$

Bob’s secret key $b$

\[ \downarrow \]

Bob’s public key $b(X, Y)$

\[ \downarrow \]

Warning: Clocks aren’t elliptic! Can attack clock cryptography, e.g., compute $a$ from public key, by combining congruences. To match RSA-3072 security need $p \approx 2^{1536}$. 
Addition on an Edwards curve

Change the curve on which Alice and Bob work.

\[ x^2 + y^2 = 1 - 30x^2y^2. \]

Sum of \((x_1, y_1)\) and \((x_2, y_2)\) is
\[
\left( \frac{x_1y_2 + y_1x_2}{1 - 30x_1x_2y_1y_2}, \frac{y_1y_2 - x_1x_2}{1 + 30x_1x_2y_1y_2} \right).
\]
The clock again, for comparison:

neutral = (0, 1)

\[ P_1 = (x_1, y_1) \]

\[ P_2 = (x_2, y_2) \]

\[ P_3 = (x_3, y_3) \]

\[ x^2 + y^2 = 1. \]

Sum of \((x_1, y_1)\) and \((x_2, y_2)\) is

\[ (x_1y_2 + y_1x_2, \quad y_1y_2 - x_1x_2). \]
“Hey, there were divisions in the Edwards addition law! What if the denominators are 0?”
Answer: They aren’t!
If $x_i = 0$ or $y_i = 0$ then
$$1 \pm 30x_1x_2y_1y_2 = 1 \neq 0.$$ 
If $x^2 + y^2 = 1 - 30x^2y^2$ then $30x^2y^2 < 1$
so $\sqrt{30} \ |xy| < 1.$
“Hey, there were divisions in the Edwards addition law! What if the denominators are 0?”
Answer: They aren’t!

If \( x_i = 0 \) or \( y_i = 0 \) then
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\]
If \( x^2 + y^2 = 1 - 30x^2 y^2 \)
then \( 30x^2 y^2 < 1 \)
so \( \sqrt{30} |xy| < 1. \)

If \( x_1^2 + y_1^2 = 1 - 30x_1^2 y_1^2 \)
and \( x_2^2 + y_2^2 = 1 - 30x_2^2 y_2^2 \)
then \( \sqrt{30} |x_1 y_1| < 1 \)
and \( \sqrt{30} |x_2 y_2| < 1 \)
“Hey, there were divisions in the Edwards addition law! What if the denominators are 0?”

Answer: They aren’t!

If $x_i = 0$ or $y_i = 0$ then
$1 \pm 30x_1x_2y_1y_2 = 1 \neq 0$.

If $x^2 + y^2 = 1 - 30x^2y^2$
then $30x^2y^2 < 1$
so $\sqrt{30} |xy| < 1$.

If $x_1^2 + y_1^2 = 1 - 30x_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 - 30x_2^2y_2^2$
then $\sqrt{30} |x_1y_1| < 1$
and $\sqrt{30} |x_2y_2| < 1$
so $30 |x_1y_1x_2y_2| < 1$
so $1 \pm 30x_1x_2y_1y_2 > 0$. 
The Edwards addition law
\[(x_1, y_1) + (x_2, y_2) = (x_3, y_3),\]
\[
\begin{align*}
(x_3, y_3) &= \left(\frac{x_1 y_2 + y_1 x_2}{1 - 30x_1 x_2 y_1 y_2}, \right. \\
&\left. \frac{y_1 y_2 - x_1 x_2}{1 + 30x_1 x_2 y_1 y_2}\right) 
\end{align*}
\]
is a group law for the curve
\[x^2 + y^2 = 1 - 30x^2 y^2.\]

Some calculation required:
addition result is on curve;
addition law is associative.

Other parts of proof are easy:
addition law is commutative;
\((0, 1)\) is neutral element;
\[(x_1, y_1) + (-x_1, y_1) = (0, 1).\]
More Edwards curves

Fix an odd prime power $q$.
Fix a non-square $d \in \mathbb{F}_q$.

\[
\{(x, y) \in \mathbb{F}_q \times \mathbb{F}_q : \quad x^2 + y^2 = 1 + dx^2 y^2\}
\]
is a commutative group with

\[(x_1, y_1) + (x_2, y_2) = (x_3, y_3)\]
defined by Edwards addition law:

\[
x_3 = \frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2},
\]
\[
y_3 = \frac{y_1 y_2 - x_1 x_2}{1 - d x_1 x_2 y_1 y_2}.
\]
Denominators are never 0.
But need different proof;
“$x^2 + y^2 > 0$” doesn’t work.
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If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$

and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$

and $dx_1 x_2 y_1 y_2 = \pm 1$
Denominators are never 0. But need different proof; “$x^2 + y^2 > 0$” doesn’t work.

\[ x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2 \]
and \[ x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2 \]
and \[ dx_1 x_2 y_1 y_2 = \pm 1 \]
then \[ dx_1^2 y_1^2 (x_2 + y_2)^2 = dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2) \]
Denominators are never 0.
But need different proof;
“\(x^2 + y^2 > 0\)” doesn’t work.

If \(x_1^2 + y_1^2 = 1 + dx_1^2y_1^2\)
and \(x_2^2 + y_2^2 = 1 + dx_2^2y_2^2\)
and \(dx_1x_2y_1y_2 = \pm 1\)
then \(dx_1^2y_1^2(x_2 + y_2)^2\)
\[= dx_1^2y_1^2(x_2^2 + y_2^2 + 2x_2y_2)\]
\[= dx_1^2y_1^2(dx_2y_2^2 + 1 + 2x_2y_2)\]
Denominators are never 0.
But need different proof; "\(x^2 + y^2 > 0\)" doesn’t work.

If \(x_1^2 + y_1^2 = 1 + dx_1^2y_1^2\)
and \(x_2^2 + y_2^2 = 1 + dx_2^2y_2^2\)
and \(dx_1x_2y_1y_2 = \pm 1\)
then \(dx_1^2y_1^2(x_2 + y_2)^2\)
\[= dx_1^2y_1^2(x_2^2 + y_2^2 + 2x_2y_2)\]
\[= dx_1^2y_1^2(dx_2^2y_2^2 + 1 + 2x_2y_2)\]
\[= d^2x_1^2y_1^2x_2^2y_2^2 + dx_1^2y_1^2 + 2dx_1^2y_1^2x_2y_2\]
Denominators are never 0. But need different proof; 
"$x^2 + y^2 > 0$" doesn’t work.

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$
then $dx_1^2 y_1^2 (x_2 + y_2)^2$
$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$
$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$
$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$
$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$
Denominators are never 0.
But need different proof;
“\(x^2 + y^2 > 0\)” doesn’t work.

If \(x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2\)
and \(x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2\)
and \(dx_1 x_2 y_1 y_2 = \pm 1\)
then \(dx_1^2 y_1^2 (x_2 + y_2)^2\)
\[= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)\]
\[= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)\]
\[= d^2 x_1^2 y_1^2 x_2^2 y_2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2\]
\[= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1\]
\[= x_1^2 + y_1^2 \pm 2x_1 y_1\]
Denominators are never 0.  
But need different proof;  
"\(x^2 + y^2 > 0\)" doesn't work. 

If \(x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2\) 
and \(x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2\) 
and \(dx_1 x_2 y_1 y_2 = \pm 1\)  
then \(dx_1^2 y_1^2 (x_2 + y_2)^2\)  
\[= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)\]  
\[= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)\]  
\[= dx_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2\]  
\[= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1\]  
\[= x_1^2 + y_1^2 \pm 2x_1 y_1\]  
\[= (x_1 \pm y_1)^2.\]
Case 1: $x_2 + y_2 \neq 0$. Then

$$d = \left( \frac{x_1 \pm y_1}{x_1 y_1 (x_2 + y_2)} \right)^2,$$

contradiction.
Case 1: \( x_2 + y_2 \neq 0 \). Then

\[
d = \left( \frac{x_1 \pm y_1}{x_1 y_1 (x_2 + y_2)} \right)^2,
\]

contradiction.

Case 2: \( x_2 - y_2 \neq 0 \). Then

\[
d = \left( \frac{x_1 \mp y_1}{x_1 y_1 (x_2 - y_2)} \right)^2,
\]

contradiction.
Case 1: \( x_2 + y_2 \neq 0 \). Then
\[
d = \left( \frac{x_1 \pm y_1}{x_1 y_1(x_2 + y_2)} \right)^2,
\]
contradiction.

Case 2: \( x_2 - y_2 \neq 0 \). Then
\[
d = \left( \frac{x_1 \mp y_1}{x_1 y_1(x_2 - y_2)} \right)^2,
\]
contradiction.

Case 3: \( x_2 + y_2 = x_2 - y_2 = 0 \). Then \( x_2 = 0 \) and \( y_2 = 0 \), contradiction.
Using ECC sensibly

Typical starting point:
Client knows secret key $a$
and server’s public key $b(X, Y)$.
Client computes (and caches) shared secret $ab(X, Y)$.

Client has packet for server.
Generates unique nonce.
Uses shared secret to encrypt and authenticate packet.

Total packet overhead:
24 bytes for nonce,
16 bytes for authenticator,
32 bytes for client’s public key.
Server receives packet, sees client’s public key $a(X, Y)$. Server computes (and caches) shared secret $ab(X, Y)$.

Server uses shared secret to verify authenticator and decrypt packet.

Client and server encrypt, authenticate, verify, and decrypt all subsequent packets in the same way, using the same shared secret.
Easy-to-use packet protection: crypto_box from nacl.cace-project.eu.

High-security curve (Curve25519). High-security implementation (e.g., no secret array indices). Extensive code validation.

Server can compute shared secrets for 1000000 new clients in 40 seconds of computation on a Core 2 Quad.

Not much hope for attacker if ECC user is running this!
Edwards curves are cool
How to compute $aP$

Use binary representation of $a$ to compute $a(X,Y)$ in $\lfloor \log_2 a \rfloor$ doublings and at most that many additions.

E.g. $a = 23 = (10111)_2$:

$$23P = 2(2(2(2P) + P) + P) + P.$$

For $a = (1, a_{n-1}, \ldots, a_1, a_0)_2$; compute scalar multiplication $aP = 2(\cdots 2(2P + a_{n-1}P) + a_{n-2}P) + \cdots + a_1P) + a_0P$.

Later: More efficient methods.
Faster group operations

Can split computation $aP$ into additions and doublings.
Formulas $(x_1, y_1) + (x_2, y_2) = ((x_1y_2 + y_1x_2)/(1 + dx_1x_2y_1y_2), (y_1y_2 - x_1x_2)/(1 - dx_1x_2y_1y_2))$ use expensive divisions.

Better: postpone divisions and work with fractions.
Represent $(x_1, y_1)$ as $(X_1 : Y_1 : Z_1)$ with $x_1 = X_1/Z_1$ and $y_1 = Y_1/Z_1$ for $Z_1 \neq 0$. 
Addition formulas in these projective coordinates:

\[ A = Z_1 \cdot Z_2; \quad B = A^2; \]
\[ C = X_1 \cdot X_2; \quad D = Y_1 \cdot Y_2; \]
\[ E = d \cdot C \cdot D; \quad F = B - E; \]
\[ G = B + E; \quad X_3 = A \cdot F \cdot ((X_1 + Y_1) \cdot (X_2 + Y_2) - C - D); \]
\[ Y_3 = A \cdot G \cdot (D - C); \quad Z_3 = F \cdot G. \]

Needs \(1S + 10M + 1M_d\).

Uses

\[(X_1 + Y_1) \cdot (X_2 + Y_2) - C - D =\]
\[X_1X_2 + X_1Y_2 + Y_1X_2 + Y_1Y_2 - X_1X_2 - Y_1Y_2 =\]
\[X_1Y_2 + Y_1X_2.\]
Doubling means $P_2 = P_1$, i.e.,

$$(x_1, y_1) + (x_1, y_1) =
((x_1y_1 + y_1x_1)/(1 + dx_1x_1y_1y_1),
(y_1y_1 - x_1x_1)/(1 - dx_1x_1y_1y_1)) =
((2x_1y_1)/(1 + dx_1^2y_1^2),
(y_1^2 - x_1^2)/(1 - dx_1^2y_1^2)).$$

Remember $P_1 = (x_1, y_1)$
is a point on the Edwards curve,
thus $x_1^2 + y_1^2 = 1 + dx_1^2y_1^2$ and
$2P_1 = ((2x_1y_1)/(x_1^2 + y_1^2),
(y_1^2 - x_1^2)/(2 - (x_1^2 + y_1^2))).$

This transformation reduced
the total degree of the equation
from 4 to 2.
Doubling formulas in projective coordinates:

\[ B = (X_1 + Y_1)^2; \quad C = X_1^2; \]
\[ D = Y_1^2; \quad E = C + D; \quad H = Z_1^2; \]
\[ J = E - 2H; \]
\[ X_3 = (B - E) \cdot J; \]
\[ Y_3 = E \cdot (C - D); \quad Z_3 = E \cdot J. \]

Needs 4S + 3M.
Doubling formulas in projective coordinates:

\[ B = (X_1 + Y_1)^2; \quad C = X_1^2; \]
\[ D = Y_1^2; \quad E = C + D; \quad H = Z_1^2; \]
\[ J = E - 2H; \]
\[ X_3 = (B - E) \cdot J; \]
\[ Y_3 = E \cdot (C - D); \quad Z_3 = E \cdot J. \]

Needs \(4S + 3M\).

1 doubling far less expensive than 1 addition.

Usual scalar multiplication uses many more doublings than additions.
More variations of addition:
e.g., in inverted coordinates \((X_1 : Y_1 : Z_1)\) corresponds to \(x_1 = Z_1/X_1\) and \(y_1 = Z_1/Y_1\).

Alternative addition formulas:
\[
(x_1, y_1) + (x_2, y_2) = \\
((x_1y_1 + x_2y_2)/(x_1x_2 + y_1y_2), \\
(x_1y_1 - x_2y_2)/(x_1y_2 - x_2y_1)).
\]

Attention: these formulas fail for doubling.

Curious fact: formulas do not involve curve parameter \(d\).
Twisted Edwards curves

Generalization to cover more curves over given finite field $F_q$:
Use $a, d \in F_q^*$ with $a \neq d$
and consider twisted Edwards curve $ax^2 + y^2 = 1 + dx^2y^2$.

Why do users want more curves?
One answer: Speed!
Particularly fast choice:
$a = -1$ gives additions in $8M$.

Another answer, coming later:
Smaller $q$ for same security.
Edwards and twisted Edwards give slight bonus to attacker.
ECDSA

Users can sign messages using Edwards curves.

Take a point $P$ on an Edwards curve modulo a prime $q > 2$.

ECDSA signer needs to know the order of $P$.

There are only finitely many other points; about $q$ in total. Adding $P$ to itself will eventually reach $(0, 1)$; let $\ell$ be the smallest integer $> 0$ with $\ell P = (0, 1)$. This $\ell$ is the order of $P$. 
The signature scheme has as system parameters a curve $E$; a base point $P$; and a hash function $h$ with output length at least $\lceil \log_2 \ell \rceil + 1$. Alice’s secret key is an integer $a$ and her public key is $P_A = aP$.

To sign message $m$, Alice computes $h(m)$; picks random $k$; computes $R = kP = (x_1, y_1)$; puts $r \equiv y_1 \mod \ell$; computes $s \equiv k^{-1}(h(m) + r \cdot a) \mod \ell$. The signature on $m$ is $(r, s)$. 
Anybody can verify signature given $m$ and $(r, s)$:
Compute $w_1 \equiv s^{-1} h(m) \mod \ell$
and $w_2 \equiv s^{-1} \cdot r \mod \ell$.
Check whether the $y$-coordinate of $w_1 P + w_2 P_A$ equals $r$ modulo $\ell$
and if so, accept signature.

Alice’s signatures are valid:

$$w_1 P + w_2 P_A = (s^{-1} h(m)) P + (s^{-1} \cdot r) P_A = (s^{-1} (h(m) + r a)) P = k P$$

and so the $y$-coordinate of this expression equals $r$,
the $y$-coordinate of $k P$. 
Attacker’s view on signatures

Anybody can produce an $R = kP$. Alice’s private key is only used in
$s \equiv k^{-1}(h(m) + r \cdot a) \mod ℓ$.

Can fake signatures if one can break the DLP, i.e., if one can compute $a$ from $P_A$.

Most of this course deals with methods for breaking DLPs. Sometimes attacks are easier...
If $k$ is known for some $m, (r, s)$
then $a \equiv (sk - h(m))/r \mod \ell$.

If two signatures $m_1, (r, s_1)$ and
$m_2, (r, s_2)$ have the same value
for $r$: assume $k_1 = k_2$; observe
$s_1 - s_2 = k_1^{-1}(h(m_1) + ra - (h(m_2) + ra))$; compute $k = (s_1 - s_2)/(h(m_1) - h(m_2))$.
Continue as above.

If bits of many $k$’s are known
(biased PRNG) can attack
$s \equiv k^{-1}(h(m) + r \cdot a) \mod \ell$
as hidden number problem
using lattice basis reduction.
Malicious signer

Alice can set up her public key so that two messages of her choice share the same signature, i.e., she can claim to have signed $m_1$ or $m_2$ at will:

$R = (x_1, y_1)$ and $-R = (-x_1, y_1)$ have the same $y$-coordinate.

Thus, $(r, s)$ fits $R = kP$,

$s \equiv k^{-1}(h(m_1) + ra) \mod \ell$ and $-R = (-k)P$,

$s \equiv -k^{-1}(h(m_2) + ra) \mod \ell$ if $a \equiv -(h(m_1) + h(m_2))/2r \mod \ell$. 
Malicious signer

Alice can set up her public key so that two messages of her choice share the same signature, i.e., she can claim to have signed $m_1$ or $m_2$ at will:

$R = (x_1, y_1)$ and $-R = (-x_1, y_1)$ have the same $y$-coordinate.

Thus, $(r, s)$ fits $R = kP$,

$s \equiv k^{-1}(h(m_1) + ra) \mod \ell$ and $-R = (-k)P$,

$s \equiv -k^{-1}(h(m_2) + ra) \mod \ell$ if $a \equiv -(h(m_1) + h(m_2))/2r \mod \ell$.

(Easy tweak: include bit of $x_1$.)
Elliptic curves

Why do we talk about Edwards curves in this course? Edwards curves are elliptic; easiest way to understand elliptic curves is Edwards.

But more elliptic curves exist! Most common representation: Weierstrass curve

$$v^2 = u^3 + a_2u^2 + a_4u + a_6.$$  

(Weierstrass has different meaning in characteristic 2 but for now we use odd char.)
Addition on Weierstrass curve

\[ v^2 = u^3 + u^2 + u + 1 \]

Slope \( \lambda = (v_2 - v_1)/(u_2 - u_1) \)
(note \( u_1 \neq u_2 \)).
Doubling on Weierstrass curve

\[ v^2 = u^3 - u \]

\[ R = -(2P) \]

Slope \( \lambda = \frac{3u_1^2 - 1}{2v_1} \).
General addition:
\[(u_P, v_P) + (u_R, v_R) = (u_{P+R}, v_{P+R}) = (\lambda^2 - u_P - u_R, \lambda(u_P - u_{P+R}) - v_P).\]

\(u_P \neq u_R, \text{ “addition”:} \)
\[\lambda = (v_R - v_P)/(u_R - u_P).\]
Total cost 1I + 2M + 1S.

\(P = R \) and \(v_P \neq 0, \text{ “doubling”:} \)
\[\lambda = (3u_P^2 + 2a_2u_P + a_4)/(2v_P).\]
Total cost 1I + 2M + 2S.

Also handle some exceptions:
\[(u_P, v_P) = (u_R, -v_R);\]
inputs at \(\infty\).
Birational equivalence

Starting from point \((x, y)\) on \(x^2 + y^2 = 1 + dx^2y^2\):

Define \(A = 2(1 + d)/(1 - d)\), \(B = 4/(1 - d)\); 
\(u = (1 + y)/(B(1 - y))\), 
\(v = u/x = (1 + y)/(Bx(1 - y))\). (Skip a few exceptional points.)

Then \((u, v)\) is a point on a Weierstrass curve:

\[v^2 = u^3 + \left(\frac{A}{B}\right)u^2 + \left(\frac{1}{B^2}\right)u.\]

Easily invert this map: 
\[x = u/v, \ y = (Bu - 1)/(Bu + 1).\]
Attacker can transform Edwards curve to Weierstrass curve and vice versa; \( n(x, y) \mapsto n(u, v) \).

\( \Rightarrow \) Same discrete-log security!

Can choose curve representation so that implementation of attack is faster/easier.

System designer can choose curve representation so that protocol runs fastest; no need to worry about security degradation.

Optimization targets are different. For now: understand designer’s choices. After that: attacks!
Faster group operations

Designer has choice of curve representation and point representation.

Most protocols do a full scalar multiplication, many doublings and additions, before they need a unique representative.

Can double and add using inversion-free systems such as projective Edwards coordinates. Faster, but more storage.
There are many perspectives on elliptic-curve computations. Early development:


1984 (published 1985) Miller, and independently


Bosma, Goldwasser–Kilian, Chudnovsky–Chudnovsky, Atkin: elliptic-curve primality proving.
The Edwards perspective is new!

1761 Euler, 1866 Gauss introduced an addition law for \( x^2 + y^2 = 1 - x^2y^2 \), the “lemniscatic elliptic curve.”

2007 Edwards generalized to many curves \( x^2 + y^2 = 1 + c^4x^2y^2 \). Theorem: have now obtained all elliptic curves over \( \overline{\mathbb{Q}} \).

2007 Bernstein–Lange: Edwards addition law is complete for \( x^2 + y^2 = 1 + dx^2y^2 \) if \( d \neq 0 \); and gives new ECC speed records.
Representing curve points

Crypto 1985, Miller, “Use of elliptic curves in cryptography”:

Given $n \in \mathbb{Z}$, $P \in E(\mathbb{F}_q)$, division-polynomial recurrence computes $nP \in E(\mathbb{F}_q)$ “in $26 \log_2 n$ multiplications”; but can do better!

“It appears to be best to represent the points on the curve in the following form:

Each point is represented by the triple $(x, y, z)$ which corresponds to the point $(x/z^2, y/z^3)$.”
1986 Chudnovsky–Chudnovsky, “Sequences of numbers generated by addition in formal groups and new primality and factorization tests”:

“The crucial problem becomes the choice of the model of an algebraic group variety, where computations mod $p$ are the least time consuming.”

Most important computations: ADD is $P, Q \rightarrow P + Q$. DBL is $P \rightarrow 2P$. 
“It is preferable to use models of elliptic curves lying in low-dimensional spaces, for otherwise the number of coordinates and operations is increasing. This limits us . . . to 4 basic models of elliptic curves.”

Short Weierstrass: 
\[ y^2 = x^3 + ax + b. \]

Jacobi intersection: 
\[ s^2 + c^2 = 1, \quad as^2 + d^2 = 1. \]

Jacobi quartic: 
\[ y^2 = x^4 + 2ax^2 + 1. \]

Hessian: 
\[ x^3 + y^3 + 1 = 3dxy. \]
Optimizing Jacobian coordinates

For “traditional” \((X/Z^2, Y/Z^3)\) on \(y^2 = x^3 + ax + b\):

1986 Chudnovsky–Chudnovsky state explicit formulas using

10\(\text{M}\) for DBL; 16\(\text{M}\) for ADD.

Consequence:

\[
\approx \left(10 \log n + 16 \frac{\log n}{\log \log n}\right) \text{M}
\]

to compute \(n, P \mapsto nP\)

using sliding-windows method of scalar multiplication.

Notation: \(\log = \log_2\).
Squaring is faster than $M$.

Here are the DBL formulas:

$$S = 4X_1 \cdot Y_1^2;$$
$$M = 3X_1^2 + aZ_1^4;$$
$$T = M^2 - 2S;$$
$$X_3 = T;$$
$$Y_3 = M \cdot (S - T) - 8Y_1^4;$$
$$Z_3 = 2Y_1 \cdot Z_1.$$

Total cost $3M + 6S + 1D$ where $S$ is the cost of squaring in $F_q$, $D$ is the cost of multiplying by $a$.

The squarings produce $X_1^2, Y_1^2, Y_1^4, Z_1^2, Z_1^4, M^2$. 
Most ECC standards choose curves that make formulas faster.

Curve-choice advice from 1986 Chudnovsky–Chudnovsky:

Can eliminate the 1D by choosing curve with \( a = 1 \).

But “it is even smarter” to choose curve with \( a = -3 \).

If \( a = -3 \) then \( M = 3\left(X_1^2 - Z_1^4\right) = 3\left(X_1 - Z_1^2\right) \cdot \left(X_1 + Z_1^2\right) \).

Replace 2S with 1M.

Now DBL costs 4M + 4S.
2001 Bernstein:

3M + 5S for DBL.

11M + 5S for ADD.

How? Easy S − M tradeoff:
instead of computing 2Y_1 \cdot Z_1,
compute (Y_1 + Z_1)^2 − Y_1^2 − Z_1^2.

DBL formulas were already computing Y_1^2 and Z_1^2.

Same idea for the ADD formulas,
but have to scale X, Y, Z

to eliminate divisions by 2.
ADD for $y^2 = x^3 + ax + b$:
$U_1 = X_1Z_2^2$, $U_2 = X_2Z_1^2$,
$S_1 = Y_1Z_2^3$, $S_2 = Y_2Z_1^3$,
many more computations.

1986 Chudnovsky–Chudnovsky:
“We suggest to write addition formulas involving $(X, Y, Z, Z^2, Z^3)$.”

Disadvantages:
Allocate space for $Z^2$, $Z^3$.
Pay $1S + 1M$ in ADD and in DBL.

Advantages:
Save $2S + 2M$ at start of ADD.
Save $1S$ at start of DBL.
1998 Cohen–Miyaji–Ono:
Store point as \((X : Y : Z)\).
If point is input to \text{ADD},
also cache \(Z^2\) and \(Z^3\).
No cost, aside from space.
If point is input to another \text{ADD},
reuse \(Z^2, Z^3\). Save \(1S + 1M\)!

Best Jacobian speeds today,
including \(S - M\) tradeoffs:
\(3M + 5S\) for \text{DBL} if \(a = -3\).
\(11M + 5S\) for \text{ADD}.
\(10M + 4S\) for \text{reADD}.
\(7M + 4S\) for \text{mADD} (i.e. \(Z_2 = 1\)).
Compare to speeds for Edwards curves \( x^2 + y^2 = 1 + dx^2y^2 \) in projective coordinates (2007 Bernstein–Lange):

- \( 3M + 4S \) for DBL.
- \( 10M + 1S + 1D \) for ADD.
- \( 9M + 1S + 1D \) for mADD.

Inverted Edwards coordinates (2007 Bernstein–Lange):

- \( 3M + 4S + 1D \) for DBL.
- \( 9M + 1S + 1D \) for ADD.
- \( 8M + 1S + 1D \) for mADD.

\[ y^2 = x^3 - 0.4x + 0.7 \]
The Weierstrass-turtle: old, trusted and slow. Warning: (picture) incomplete!
\[ x^2 + y^2 = 1 - 300x^2y^2 \]
The Edwards starfish: new, fast and complete!
Start!
Weierstrass sets off. Edwards
left behind sleeping
Weierstrass has made some progress - finally Edwards wakes up.
Exciting progress: Edwards about to overtake!!
And the winner is: Edwards!
Speed-oriented Jacobian standards

2000 IEEE “Std 1363” uses Weierstrass curves in Jacobian coordinates to “provide the fastest arithmetic on elliptic curves.” Also specifies a method of choosing curves $y^2 = x^3 - 3x + b$.

2000 NIST “FIPS 186–2” standardizes five such curves.

2005 NSA “Suite B” recommends two of the NIST curves as the only public-key cryptosystems for U.S. government use.
Projective for Weierstrass

1986 Chudnovsky–Chudnovsky: Speed up ADD by switching from \((X/Z^2, Y/Z^3)\) to \((X/Z, Y/Z)\). 
\(7M + 3S\) for DBL if \(a = -3\). 
\(12M + 2S\) for ADD. 
\(12M + 2S\) for reADD.

Option has been mostly ignored: DBL dominates in ECDH etc. But ADD dominates in some applications: e.g., batch signature verification.
Montgomery curves

1987 Montgomery:

Use \( by^2 = x^3 + ax^2 + x \).

Choose small \((a + 2)/4\).

\[
2(x_2, y_2) = (x_4, y_4)
\]

\[
\Rightarrow x_4 = \frac{(x_2^2 - 1)^2}{4x_2(x_2^2 + ax_2 + 1)}.
\]

\[
(x_3, y_3) - (x_2, y_2) = (x_1, y_1),
\]

\[
(x_3, y_3) + (x_2, y_2) = (x_5, y_5)
\]

\[
\Rightarrow x_5 = \frac{(x_2x_3 - 1)^2}{x_1(x_2 - x_3)^2}.
\]
Represent \((x, y)\) as \((X:Z)\) satisfying \(x = X/Z\).

\[
B = (X_2 + Z_2)^2,
\]
\[
C = (X_2 - Z_2)^2,
\]
\[
D = B - C, \quad X_4 = B \cdot C,
\]
\[
Z_4 = D \cdot (C + D(a + 2)/4) \Rightarrow
\]
\[
2(X_2:Z_2) = (X_4:Z_4).
\]

\[
(X_3:Z_3) - (X_2:Z_2) = (X_1:Z_1),
\]
\[
E = (X_3 - Z_3) \cdot (X_2 + Z_2),
\]
\[
F = (X_3 + Z_3) \cdot (X_2 - Z_2),
\]
\[
X_5 = Z_1 \cdot (E + F)^2,
\]
\[
Z_5 = X_1 \cdot (E - F)^2 \Rightarrow
\]
\[
(X_3:Z_3) + (X_2:Z_2) = (X_5:Z_5).
\]
This representation does not allow ADD but it allows DADD, “differential addition”: 
\[ Q, R, Q - R \mapsto Q + R. \]

e.g. \[ 2P, P, P \mapsto 3P. \]
e.g. \[ 3P, 2P, P \mapsto 5P. \]
e.g. \[ 6P, 5P, P \mapsto 11P. \]

\[ 2M + 2S + 1D \] for DBL.
\[ 4M + 2S \] for DADD.
Save \[ 1M \] if \( Z_1 = 1. \)

Easily compute \( n(X_1 : Z_1) \) using \( \approx \lg n \) DBL, \( \approx \lg n \) DADD.
Almost as fast as Edwards \( nP. \)
Relatively slow for \( mP + nQ \) etc.
Doubling-oriented curves

2006 Doche–Icart–Kohel:

Use $y^2 = x^3 + ax^2 + 16ax$.
Choose small $a$.

Use $(X : Y : Z : Z^2)$
to represent $(X/Z, Y/Z^2)$.

$3M + 4S + 2D$ for DBL.

How? Factor DBL as $\hat{\phi}(\varphi)$
where $\varphi$ is a 2-isogeny.

2007 Bernstein–Lange:

$2M + 5S + 2D$ for DBL
on the same curves.
$12M + 5S + 1D$ for ADD.
Slower ADD than other systems, typically outweighing benefit of the very fast DBL.

But isogenies are useful.
Example, 2005 Gaudry:
fast DBL + DADD on Jacobians of genus-2 hyperelliptic curves, using similar factorization.

Tricky but potentially helpful:
tripling-oriented curves (see 2006 Doche–Icart–Kohel),
double-base chains, . . .
Hessian curves

Credited to Sylvester by 1986 Chudnovsky–Chudnovsky:

\((X : Y : Z)\) represent \((X/Z, Y/Z)\) on \(x^3 + y^3 + 1 = 3dxy\).

\(12M\) for ADD:

\[
\begin{align*}
X_3 &= Y_1 X_2 \cdot Y_1 Z_2 - Z_1 Y_2 \cdot X_1 Y_2, \\
Y_3 &= X_1 Z_2 \cdot X_1 Y_2 - Y_1 X_2 \cdot Z_1 X_2, \\
Z_3 &= Z_1 Y_2 \cdot Z_1 X_2 - X_1 Z_2 \cdot Y_1 Z_2.
\end{align*}
\]

\(6M + 3S\) for DBL.
2001 Joye–Quisquater:
\[ 2(X_1 : Y_1 : Z_1) = (Z_1 : X_1 : Y_1) + (Y_1 : Z_1 : X_1) \]
so can use ADD to double.

“Unified addition formulas,” helpful against side channels.
But need to permute inputs.

2009 Bernstein–Kohel–Lange: Easily avoid permutation!

2008 Hisil–Wong–Carter–Dawson:
\[ (X : Y : Z : X^2 : Y^2 : Z^2 : 2XY : 2XZ : 2YZ) \]
\[ 6M + 6S \text{ for ADD.} \]
\[ 3M + 6S \text{ for DBL.} \]
\[ x^3 - y^3 + 1 = 0.3xy \]
The Hessian-ray: uniform

but

not strongly so
Jacobi intersections

1986 Chudnovsky–Chudnovsky:

\((S : C : D : Z)\) represent

\((S/Z, C/Z, D/Z)\) on

\(s^2 + c^2 = 1, \quad a s^2 + d^2 = 1.\)

14M + 2S + 1D for ADD.

“Tremendous advantage” of being strongly unified.

5M + 3S for DBL.

“Perhaps (?) . . . the most efficient duplication formulas which do not depend on the coefficients of an elliptic curve.”
2001 Liardet–Smart:
\[13M + 2S + 1D\text{ for ADD.}\]
\[4M + 3S\text{ for DBL.}\]

2007 Bernstein–Lange:
\[3M + 4S\text{ for DBL.}\]

2008 Hisil–Wong–Carter–Dawson:
\[13M + 1S + 2D\text{ for ADD.}\]
\[2M + 5S + 1D\text{ for DBL.}\]
Also \((S : C : D : Z : SC : DZ)\):
\[11M + 1S + 2D\text{ for ADD.}\]
\[2M + 5S + 1D\text{ for DBL.}\]
Jacobi quartics

\((X:Y:Z)\) represent \((X/Z, Y/Z^2)\) on \(y^2 = x^4 + 2ax^2 + 1\).

1986 Chudnovsky–Chudnovsky:

\(3M + 6S + 2D\) for DBL.

Slow ADD.

2002 Billet–Joye:

New choice of neutral element.

\(10M + 3S + 1D\) for ADD, strongly unified.

2007 Bernstein–Lange:

\(1M + 9S + 1D\) for DBL.
2007 Hisil–Carter–Dawson: 
$2\textbf{M} + 6\textbf{S} + 2\textbf{D}$ for DBL.

2007 Feng–Wu: 
$2\textbf{M} + 6\textbf{S} + 1\textbf{D}$ for DBL. 
$1\textbf{M} + 7\textbf{S} + 3\textbf{D}$ for DBL

on curves chosen with $a^2 + c^2 = 1$.

More speedups: 2007 Duquesne, 
2007 Hisil–Carter–Dawson, 
2008 Hisil–Wong–Carter–Dawson: 
use $(X : Y : Z : X^2 : Z^2)$ 
or $(X : Y : Z : X^2 : Z^2 : 2XZ)$.

Can combine with Feng–Wu. 
Competitive with Edwards!
\[ x^2 = y^4 - 1.9y^2 + 1 \]
The Jacobi-quartic squid: can be extended to XXYZZR giant squid.
Feb
More addition formulas

Explicit-Formulas Database: hyperelliptic.org/EFD

EFD has 581 computer-verified formulas and operation counts for ADD, DBL, etc. in 51 representations on 13 shapes of elliptic curves.

Not yet handled by computer: generality of curve shapes (e.g., Hessian order $\in 3\mathbb{Z}$); complete addition algorithms (e.g., checking for $\infty$).
Notation

If

\[ n = \sum_{i=0}^{l-1} n_i 2^i \]

we write \( n \) in binary representation

\[ n = (n_{l-1} \ldots n_0)_2. \]

E.g. \( n = 35 = 32 + 2 + 1 = 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0, \)
then \( 35 = (100011)_2. \)

The following algorithms are stated in some group \((G, +)\) with neutral element \( O \). Scalar multiplication is denoted by

\[ nP = P + P + \cdots + P \ (n \text{ terms}). \]

Right-to-Left Binary

**IN:** An element $P \in G$
and a positive integer $n = (n_{l-1} \ldots n_0)_2$.

**OUT:** The element $nP \in G$.

$R \leftarrow O$, $Q \leftarrow P$,
for $i = 0$ to $l - 2$ do
  if $n_i = 1$ then $R \leftarrow P + Q$
  $Q \leftarrow 2Q$
if $n_{l-1} = 1$ then $R \leftarrow P + Q$
return $R$

This algorithm computes $35P = 2^5P + 2^1P + P$.
At the end of step 2, $Q = 2^{i+1}P$
and $R = (n_i \ldots n_0)_2P$. 
Left–to–Right Binary

**IN:** An element \( P \in G \) and a positive integer \( n = (n_{l-1} \ldots n_0)_2, n_{l-1} = 1 \).

**OUT:** The element \( nP \in G \).

\[
\begin{align*}
R & \leftarrow P \\
\text{for } i = l - 2 \text{ to } 0 \text{ do} \\
& \quad R \leftarrow 2R \\
& \quad \text{if } n_i = 1 \text{ then } R \leftarrow P + R \\
\text{return } R
\end{align*}
\]

This algorithm computes \( 35P = 2(2(2(2(2P)))) + P) + P \).

The intermediate variable \( R \) holds \( (n_{l-1} \ldots n_i)_2P \).
Number of additions

For each 1 in the binary representation of $n$ we compute an addition. On average there are $l/2$ non-zero coefficients.

In some groups (e.g. elliptic curves) $P + Q$ has the same cost as $P - Q$, so it makes sense to use negative coefficients. This gives signed binary expansions.

Note that $31 = 2^4 + 2^3 + 2^2 + 2 + 1 = 2^5 - 1$ and so $31 P = 2(2(2(2P + P) + P) + P) + P)$
$$= 2(2(2(2(2P)))))) - P$$
Can always replace two adjacent 1’s in the binary expansion by 10\(\bar{1}\) since \((11)_2 = (10\bar{1})_s\). (\(\bar{1}\) denotes \(-1\)).

By systematically replacing runs of 1’s we can guarantee that there are no two adjacent bits that are nonzero.

A representation fulfilling this is called a “non-adjacent form” (NAF).
NAF’s have the lowest density among all signed binary expansions (with coefficients in \{0, 1, -1\}).

\[(10010100110111010\underline{110})_2 = (10010100110111010\underline{110}\overline{10})_2 = (10010100110111110\overline{10}\overline{10})_2 = (10010100111000\overline{10}\overline{10}\overline{10})_2 = (1001010100\overline{10}00\overline{10}\overline{10}\overline{10})_2\]

Results no worse, but not necessarily better:
\[35 = (100011)_2 = (10010\overline{1})_s.\]
Non-Adjacent Form

IN: Positive integer \( n = (n_l n_{l-1} \ldots n_0)_2 \), \( n_l = n_{l-1} = 0 \).

OUT: NAF of \( n \), \( (n'_{l-1} \ldots n'_0)_s \).

\[ c_0 \leftarrow 0 \]

\[ \text{for } i = 0 \text{ to } l - 1 \text{ do} \]

\[ c_{i+1} \leftarrow \lceil (c_i + n_i + n_{i+1})/2 \rceil \]

\[ n'_i \leftarrow c_i + n_i - 2c_{i+1} \]

\[ \text{return } (n'_{l-1} \ldots n'_0)_s \]

Resulting signed binary expansion has length at most \( l + 1 \), so longer by at most 1 bit.

On average there are \( l/3 \) non-zero coefficients.

Can also do left-to-right.
NAF – example

\[ c_0 \leftarrow 0 \]

for \( i = 0 \) to \( l - 1 \) do

\[ c_{i+1} \leftarrow \lfloor (c_i + n_i + n_{i+1})/2 \rfloor \]

\[ n'_i \leftarrow c_i + n_i - 2c_{i+1} \]

return \((n'_l \ldots n'_0)_s\)

\[ 35 = (00100011)_2, \quad c_0 = 0 \]

\[ c_1 = \lfloor (0 + 1 + 1)/2 \rfloor = 1, \]

\[ n_0 = 0 + 1 - 2 = -1 \]

\[ c_2 = \lfloor (1 + 1 + 0)/2 \rfloor = 1, \]

\[ n_1 = 1 + 1 - 2 = 0 \]

\[ c_3 = \lfloor (1 + 0 + 0)/2 \rfloor = 0, \]

\[ n_2 = 1 + 0 - 0 = 1 \]
\[ c_4 = \left[ \frac{(0 + 0 + 0)}{2} \right] = 0, \]
\[ n_3 = 0 + 0 - 0 = 0 \]
\[ c_5 = \left[ \frac{(0 + 0 + 1)}{2} \right] = 0, \]
\[ n_4 = 0 + 0 - 0 = 0 \]
\[ c_6 = \left[ \frac{(0 + 1 + 0)}{2} \right] = 0, \]
\[ n_5 = 0 + 1 - 0 = 1 \]
\[ c_7 = \left[ \frac{(0 + 0 + 0)}{2} \right] = 0, \]
\[ n_6 = 0 + 0 - 0 = 0 \]
\Rightarrow 35 = (10010\bar{1})_s
Generalizations

So far all expansions in base 2 (signed or unsigned).

Generalize to larger base; often $2^w$ ($w > 1$). Then the coefficients are in $[0, 2^w - 1]$. Also fractional windows have been suggested.

$w$ is called window width.

Assume that $mP$ for $m \in [0, 2^w - 1]$ are precomputed.

Easiest way: just group $w$ bits.
Sliding windows: Group \( w \) bits and skip forward if LSB is 0 (requires only odd integers in \([0, 2^w - 1]\)) as coefficients and leads to \( l/(w + 1) \) additions).

If \( - \) is cheap, use signed sliding windows; this leads to \( l/(w + 2) \) additions.

\[
\begin{align*}
(10010100110111010110)_{2} &= (02010100030103010102)_{2} = (2110313112)_{4},
\end{align*}
\]
needs 8 additions and precomputed \( 2P \) and \( 3P \).
\[(100101001101110110110)_{2} = (10010100030103010030)_{2},\]
needs 7 additions and only precomputed 3P.

\[(10010100110111010110)_{2} = (100101001110000\bar{3}0030)_{2} = (1001010100\bar{1}0000\bar{3}0030)_{2} = (1001100\bar{3}00\bar{1}0000\bar{3}0030)_{2} = (1000300\bar{3}00\bar{1}0000\bar{3}0030)_{2}\]
needs 5 additions and precomputed 3P, assuming that — is available.
Montgomery Ladder

Consider the intermediate results ($i$ is decreasing).

\[ Q_i = \sum_{j=i}^{l} n_j 2^{j-i} P, \text{ put } R_i = Q_i + P, \text{ then } Q_i = 2Q_{i+1} + n_i P = Q_{i+1} + R_{i+1} + n_i P - P = 2R_{i+1} + n_i P - 2P. \]

This implies \((Q_i, R_i) = \)

\[
\begin{cases} 
(2Q_{i+1}, Q_{i+1} + R_{i+1}) & \text{if } n_i = 0 \\
(Q_{i+1} + R_{i+1}, 2R_{i+1}) & \text{if } n_i = 1
\end{cases}
\]

13 = (1101)\(_2\):

\[
\begin{align*}
(Q_3, R_3) &= (P, 2P) \\
(Q_2, R_2) &= (3P, 4P) \\
(Q_1, R_1) &= (6P, 7P) \\
(Q_0, R_0) &= (13P, 14P)
\end{align*}
\]
Idea of joint doublings

To compute
\[ n_1 P_1 + n_2 P_2 + \cdots + n_m P_m \]
compute the doublings together, i.e. write scalars \( n_i \) in binary:
\[
n_1 = n_{1,l-1}2^{l-1} + n_{1,l-2}2^{l-2} + \cdots + n_{1,0} \\
n_2 = n_{2,l-1}2^{l-1} + n_{2,l-2}2^{l-2} + \cdots + n_{2,0} \\
\vdots \quad = \quad \vdots \quad \vdots \\
n_m = n_{m,l-1}2^{l-1} + n_{m,l-2}2^{l-2} + \cdots + n_{m,0}
\]
Idea of joint doublings

To compute
\[ n_1 P_1 + n_2 P_2 + \cdots + n_m P_m \]
compute the doublings together, i.e. write scalars \( n_i \) in binary:

\[
\begin{align*}
n_1 &= n_{1,l-1} 2^{l-1} + n_{1,l-2} 2^{l-2} + \cdots + n_{1,0} \\
n_2 &= n_{2,l-1} 2^{l-1} + n_{2,l-2} 2^{l-2} + \cdots + n_{2,0} \\
\vdots &= \vdots \\
n_m &= n_{m,l-1} 2^{l-1} + n_{m,l-2} 2^{l-2} + \cdots + n_{m,0}
\end{align*}
\]

Compute as
\[
2 \left( 2(n_{1,l-1} P_1 + n_{2,l-1} P_2 + \cdots + n_{m,l-1} P_m) + (n_{1,l-2} P_1 + n_{2,l-2} P_2 + \cdots + n_{m,l-2} P_m) \right) + \cdots
\]

etc. Even with precomputations, many more adds than doublings.
Applications

ECDSA verification uses 2 scalar multiplications ... just to add the results.

If base point $P$ is fixed, precompute $R = 2^{l/2}P$ and include in the curve parameters. Split scalar $n = n_12^{l/2} + n_0$ and compute

$$n_1R + n_0P.$$ 

GLV curves split scalar in two halves to get faster scalar multiplication.
Verification in accelerated ECDSA can be extended to use 4 or even 6 scalars. Splitting of the scalar is done by LLL techniques.

Further applications in batch verification of signatures—many scalars—by taking random linear combinations.
Examples

$$1338P + 1715Q = (10100111010)_2 P + (11010110011)_2 Q$$

takes 20 doublings and 12 additions.

Given precomputed $P + Q$, we can compute the same in 10 doublings and 8 additions.

If $-$ is efficient

$$(01010100\bar{1}010)_2 P + (100\bar{1}0\bar{1}0\bar{1}01\bar{1})_2 Q$$

reduces (compared to first line) the number of additions to 10, but needs 21 doublings.
Given precomputed $P + Q$ and $P - Q$:

$$(01010100\overline{1}010)_{2}P + (100\overline{1}0\overline{1}0101\overline{1})_{2}Q$$

needs 11 doublings and 8 additions.

The joint Hamming weight has not decreased, and length has increased.
Joint-Sparse-Form

Solinas introduced generalization of NAF to two scalars:

Of any three consecutive columns, at least one is a zero column, that is for all \( i \) and all positive \( j > 1 \) one has \( n_{i,j+k} = n_{1-i,j+k} = 0 \) for at least one \( k \) in \( \{0, \pm 1\} \).

Adjacent terms do not have opposite signs, i.e. it is never the case that \( n_{i,j} n_{i,j+1} = -1 \).

If \( n_{i,j+1} n_{i,j} \neq 0 \) then one has \( n_{1-i,j+1} = \pm 1 \) and \( n_{1-i,j} = 0 \).
1338 =
(10100111010)_2 =
(01010100\overline{1}010)_2
1715 =
(11010110011)_2 =
(100\overline{1}0\overline{1}0\overline{1}0011)_2

So $1338P + 1715Q$ needs 7 additions and 11 doublings given precomputed $P + Q$ and $P - Q$. Average joint density is $1/2$, i.e. for two integers of length $l$ (each) we need on average $l$ doublings and $l/2$ additions.
Joint sparse form recoding

**IN:** Nonnegative $l$-bit integers $n_0$ and $n_1$ not both zero.

**OUT:** The joint sparse form of $n_0$ and $n_1$.

$j \leftarrow 0$, $d_0 \leftarrow 0$ and $d_1 \leftarrow 0$

while $n_0 + d_0 > 0$ OR $n_1 + d_1 > 0$
do

$l_0 \leftarrow d_0 + n_0$

$l_1 \leftarrow d_1 + n_1$

for $i = 0$ to $1$
do

if $l_i$ is even

then $n_{i,j} \leftarrow 0$

else $n_{i,j} \leftarrow l_i \text{ mods } 4$

if $l_i \equiv \pm 3 \pmod{8}$
AND \( l_{1-i} \equiv 2 \pmod{4} \)

then \( n_{i,j} \leftarrow -n_{i,j} \)

for \( i = 0 \) to \( 1 \) do

if \( 2d_i = 1 + n_{i,j} \)

then \( d_i \leftarrow 1 - d_i \)

\( n_i \leftarrow \lfloor n_i/2 \rfloor \)

\( j \leftarrow j + 1 \)

return \((n_{0,l} \ldots n_{0,0})_s\)

and \((n_{1,l} \ldots n_{1,0})_s\)
Asymptotic speeds

1939 Brauer algorithm:

\[ \approx (1 + 1/ \lg \lg H) \lg H \]

additions to compute

\[ P \mapsto nP \text{ if } n \leq H. \]

1964 Straus algorithm:

\[ \approx (1 + k/ \lg \lg H) \lg H \]

additions to compute

\[ P_1, \ldots, P_k \mapsto n_1 P_1 + \cdots + n_k P_k \]

if \( n_1, \ldots, n_k \leq H. \)
1976 Yao algorithm:
\[ \approx (1 + k / \lg \lg H) \lg H \]
additions to compute
\[ P \mapsto n_1 P, \ldots, n_k P \]
if \( n_1, \ldots, n_k \leq H \).

1976 Pippenger algorithm:
Similar asymptotics,
but replace \( \lg \lg H \) with \( \lg(k \lg H) \).
Faster than Straus and Yao
if \( k \) is large.
(Knuth summary is wrong.)
More generally, Pippenger’s algorithm computes \( \ell \) sums of multiples of \( k \) inputs.

\[
\approx \left( \min\{k, \ell\} + \frac{kl}{\lg(k\ell \lg H)} \right) \lg H
\]

additions

if all coefficients are below \( H \).

Within \( 1 + \epsilon \) of optimal.
More generally, Pippenger’s algorithm computes \( \ell \) sums of multiples of \( k \) inputs.

\[
\approx \left( \min\{k, \ell\} + \frac{kl}{\lg(k\ell \lg H)} \right) \lg H
\]

additions

if all coefficients are below \( H \).

Within \( 1 + \epsilon \) of optimal.

Various special cases of Pippenger’s algorithm were reinvented and patented by 1993 Brickell–Gordon–McCurley–Wilson, 1995 Lim–Lee, etc.

Is that the end of the story?
No! 1989 Bos–Coster:

If \( n_1 \geq n_2 \geq \cdots \) then

\[
\begin{align*}
n_1 P_1 + n_2 P_2 + n_3 P_3 + \cdots &= (n_1 - q n_2) P_1 + n_2 (q P_1 + P_2) + n_3 P_3 + \cdots \\
\text{where } q &= \lfloor n_1 / n_2 \rfloor.
\end{align*}
\]

Remarkably simple; competitive with Pippenger for random choices of \( n_i \)'s; much better memory usage.
Example of Bos–Coster:

000100000 = 32
000010000 = 16
100101100 = 300
010010010 = 146
001001101 = 77
000000010 = 2
000000001 = 1

Reduce largest row:

000100000 = 32
000010000 = 16
010011010 = 154 ←
010010010 = 146
001001101 = 77
000000010 = 2
000000001 = 1

Reduce largest row:

000100000 = 32
000010000 = 16
000001000 = 8 ←
010010010 = 146
001001101 = 77
000000010 = 2
000000001 = 1

plus 2 additions.
Reduce largest row:

\[ 000100000 = 32 \]
\[ 000010000 = 16 \]
\[ 000001000 = 8 \]
\[ 001000101 = 69 \]
\[ 001001101 = 77 \]
\[ 000000010 = 2 \]
\[ 000000001 = 1 \]

plus 3 additions.
Reduce largest row:

000100000 = 32
000010000 = 16
000001000 = 8
001000101 = 69
000001000 = 8 ←
000000010 = 2
000000001 = 1

plus 4 additions.
Reduce largest row:

000100000 = 32
000010000 = 16
000001000 = 8
000100101 = 37 ←
000001000 = 8
000000010 = 2
000000001 = 1

plus 5 additions.
Reduce largest row:

000100000 = 32
000010000 = 16
000001000 = 8
000000101 = 5 ←
000001000 = 8
000000010 = 2
000000001 = 1

plus 6 additions.
Reduce largest row:

\[000010000 = 16 \leftarrow\]
\[000010000 = 16\]
\[000001000 = 8\]
\[000000101 = 5\]
\[000001000 = 8\]
\[000001000 = 8\]
\[000000010 = 2\]
\[000000001 = 1\]

plus 7 additions.
Reduce largest row:

0000000000 = 0
0000100000 = 16
0000010000 = 8
0000001011 = 5
0000010000 = 8
0000000100 = 2
0000000001 = 1

plus 7 additions.
Reduce largest row:

0000000000 = 0
0000010000 = 8 ←
0000010000 = 8
0000001011 = 5
0000010000 = 8
0000000100 = 2
0000000001 = 1

plus 8 additions.
Reduce largest row:

0000000000 = 0
0000000000 = 0 ←
0000010000 = 8
0000001010 = 5
0000010000 = 8
0000000100 = 2
0000000001 = 1

plus 8 additions.
Reduce largest row:

000000000 = 0
000000000 = 0
000000000 = 0 ←
000000101 = 5
000001000 = 8
000000100 = 8
000000010 = 2
000000001 = 1

plus 8 additions.
Reduce largest row:

0000000000 = 0
0000000000 = 0
0000000000 = 0
000000101 = 5
000000011 = 3 ←
000000010 = 2
000000001 = 1

plus 9 additions.
Reduce largest row:

0000000000 = 0
0000000000 = 0
0000000000 = 0
0000000100 = 2 ←
0000000111 = 3
0000000100 = 2
0000000011 = 1

plus 10 additions.
Reduce largest row:


\[\begin{align*}
000000000 & = 0 \\
000000000 & = 0 \\
000000000 & = 0 \\
000000010 & = 2 \\
000000001 & = 1 & \leftarrow \\
000000010 & = 2 \\
000000001 & = 1 \\
\end{align*}\]

plus 11 additions.
Reduce largest row:

0000000000 = 0
0000000000 = 0
0000000000 = 0
0000000000 = 0 ←
000000001 = 1
000000010 = 2
000000001 = 1

plus 11 additions.
Reduce largest row:

0000000000 = 0
0000000000 = 0
0000000000 = 0
0000000000 = 0
0000000001 = 1
0000000001 = 1
0000000001 = 1 ←
0000000001 = 1

plus 12 additions.
Reduce largest row:

0000000000 = 0
0000000000 = 0
0000000000 = 0
0000000000 = 0
0000000000 = 0
0000000000 = 0
0000000000 = 0 ←
0000000001 = 1
0000000001 = 1

plus 12 additions.
Reduce largest row:

\[
\begin{align*}
000000000 & = 0 \\
000000000 & = 0 \\
000000000 & = 0 \\
000000000 & = 0 \\
000000000 & = 0 \\
000000000 & = 0 \\
000000000 & = 0 \\
000000000 & = 0 \leftarrow \\
0000000001 & = 1
\end{align*}
\]

plus 12 additions.
Reduce largest row:

\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]
\[000000000 = 0\]

plus 12 additions.

Final addition chain: 1, 2, 3, 5, 8, 16, 32, 37, 69, 77, 146, 154, 300.

Short, no temporary storage, low two-operand complexity.