

## Cryptography I, homework sheet 4

Due: 03 October 2013, 10:45

Please submit your homework electronically; the TAs do not want to receive homework on paper. Please bundle your scans into one pdf file. Team up in groups of two or three to hand in your homework. We do not have capacity to correct all homeworks individually. To submit your homework, email it to `crypto13@tue.nl`.

In general, you may use computer algebra systems such as mathematica and sage; please submit your code as part of your homework if you end up using a system. Accepted systems/languages: Sage, mathematica, matlab, Pari-GP, Java.

This time one-line answers using a computer algebra system do *not* count. But it is a good moment to familiarize yourself with some system(s) so that you know how to solve similar problems for real life examples and to verify your answers. You may use a computer algebra system to compute subresults, such as  $f \operatorname{div} g$ .

1. Compute the extended gcd of 155 and 649 using Algorithm 1.
2. Compute the extended gcd of 5007 and 6891 using Algorithm 1.
3. Compute the extended gcd of  $f(x) = x^5 + 3x^3 + x^2 + 2x + 1$  and  $g(x) = x^4 - 5x^3 - 5x^2 - 5x - 6$  in  $\mathbb{Q}[x]$  using Algorithm 1.
4. Compute the extended gcd of  $f(x) = x^{11} + x^9 + x^7 + x^4 + x^3 + x + 1$  and  $g(x) = x^8 + x^5 + x^4 + x^3 + x + 1$  in  $\mathbb{F}_2[x]$  using Algorithm 1.

The rest is for your entertainment:

Here is one version of the extended Euclidean (XGCD) algorithm. This description assumes that the input elements  $f, g$  live in some ring  $R$  in which the greatest common divisor is defined. We will usually use the XGCD on integers or polynomials. If the inputs are integers you can ignore the part the leading coefficient.

### Algorithm 1 (Extended Euclidean algorithm)

IN:  $f, g \in R$

OUT:  $d, u, v \in R$  with  $d = uf + vg$

1.  $a \leftarrow [f, 1, 0]$
2.  $b \leftarrow [g, 0, 1]$
3. **repeat**
  - (a)  $c \leftarrow a - (a[1] \operatorname{div} b[1])b$
  - (b)  $a \leftarrow b$
  - (c)  $b \leftarrow c$
- while**  $b[1] \neq 0$
4.  $l \leftarrow LC(a[1])$ ,  $a \leftarrow a/l$  /\*LC = leading coefficient, this only applies to polynomials\*/
5.  $d \leftarrow a[1]$ ,  $u \leftarrow a[2]$ ,  $v \leftarrow a[3]$
6. **return**  $d, u, v$

In this algorithm,  $\text{div}$  denotes division with remainder. The first component of  $c$  is thus easier written as  $c[1] \leftarrow a[1] \bmod b[1]$  but by operating on the whole vector we get to update the values leading to  $u$  and  $v$ , too. At each step we have

$$a[1] = a[2]f + a[3]g \text{ and } b[1] = b[2]f + b[3]g.$$

To see this, note that this holds trivially for the initial conditions. If it holds for both  $a$  and  $b$  then also for  $c$  since it computes a linear relation of both vectors. So each update maintains the relation and eventually when  $b[1] = 0$ , we have that  $a[1]$  holds the previous remainder, which is the gcd of  $f$  and  $g$ . If the inputs are polynomials, at the end the gcd is made monic by dividing by the leading coefficient  $LC(a[1])$ .

**Example 2** Let  $R = \mathbb{R}[x]$  and  $f(x) = x^5 + 3x^3 - x^2 - 4x + 1$ ,  $g(x) = x^4 - 8x^3 + 8x^2 + 8x - 9$ . So at first we have  $a = [f, 1, 0]$ ,  $b = [g, 0, 1]$ .

We have  $(a[1] \text{ div } b[1]) = x + 8$  and so end the first round with

$$\begin{aligned} a &= [g, 0, 1], \\ b &= [59x^3 - 73x^2 - 59x + 73, 1, -x - 8]. \end{aligned}$$

Indeed  $b[1] = f(x) + (-x - 8)g(x)$ .

With these new values we have  $(a[1] \text{ div } b[1]) = 1/59x - 399/3481$  and so the second round ends with

$$\begin{aligned} a &= [59x^3 - 73x^2 - 59x + 73, 1, -x - 8], \\ b &= [2202/3481x^2 - 2202/3481, -1/59x + 399/3481, 1/59x^2 + 73/3481x + 289/3481]. \end{aligned}$$

In the third round we have  $(a[1] \text{ div } b[1]) = 205379/2202x - 254113/2202$  and obtain

$$\begin{aligned} a &= [2202/3481x^2 - 2202/3481, -1/59x + 399/3481, 1/59x^2 + 73/3481x + 289/3481], \\ b &= [0, 3481/2202x^2 - 13924/1101x + 10443/734, -3481/2202x^3 - 6962/1101x + 3481/2202]. \end{aligned}$$

Since  $b[1] = 0$  the loop terminates. We have  $LC(a[1]) = 2202/3481$  and thus normalize to

$$a = [x^2 - 1, -59/2202x + 133/734, 59/2202x^2 + 73/2202x + 289/2202].$$

We check that indeed

$$\begin{aligned} x^2 - 1 &= (-59/2202x + 133/734)(x^5 + 3x^3 - x^2 - 4x + 1) + \\ &\quad (59/2202x^2 + 73/2202x + 289/2202)(x^4 - 8x^3 + 8x^2 + 8x - 9). \end{aligned}$$