

Explanation:

Miller Rabin: How to apply?

Check primality of  $n$  where  $n-1=2^r \cdot t$ ,  $t$  odd

1. Pick random  $a > 0$
2. Compute  $b \equiv a^t \pmod n$  (congruence)
3. If  $b \in \{-1, 1\}$  then "probably prime"
4. For  $i=1$  to  $r-1$  do
  - a) compute  $b \equiv b^2 \pmod n$  (assigning to  $b$  the new value)
  - b) if  $b \equiv -1$  output "probably prime"
  - c) if  $b \equiv 1$  output "n not prime"
5. output "n not prime"

Iterate this for  $l$  choices of  $a$  to get probability of  $2^{l-1}$ .

Why does it work?

Fermat says  $a^{(n-1)} \equiv a^{(t \cdot 2^r)} \equiv 1 \pmod n$  if  $n$  is prime

so in the final squaring we need to reach 1 or  $n$  is not prime.

If  $n$  is prime then there are 2 square roots of 1, namely 1 and -1.

If  $n = p \cdot q$  then there are 4 roots, for  $k$  different factors there are  $2^k$  roots, because of CRT.

$$x^2 \equiv 1 \pmod n, \text{ let } n = p \cdot q.$$

$$x^2 \equiv 1 \pmod p$$

$$x^2 \equiv 1 \pmod q$$

$p$  and  $q$  are primes, so there 2 squareroots This gives 4 different CRT systems

$$x \equiv \pm 1 \pmod p$$

$$x \equiv \pm 1 \pmod q$$

with signs taken independently, these give 4 different solutions, namely

$x \equiv 1 \pmod n$  for both choices  $+$ ,  $x \equiv -1 \pmod n$  if both choices are  $-$  and a different solution  $x \equiv c \pmod n$  in the case of  $+$  for  $p$ ,  $-$  for  $q$  and  $-c$  in the other.

If we find  $c$  with  $c^2 \equiv 1 \pmod n$  and  $c$  is not  $\pm 1$  then  $n$  cannot be prime.

Miller Rabin tries to find such  $c$ , knowing that  $a^{(n-1)} \equiv 1$ , and we can compute  $r$  square roots of that -- by building the powers of  $a^t$  by squaring. If Fermat holds, we must compute 1 eventually, and if  $n$  is prime, we must have encountered -1 before that.

This covers the for loop, if  $a^t$  is already 1 then we don't get any information.