Linear feedback shift registers

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Eindhoven University of Technology

2WF80: Introduction to Cryptology

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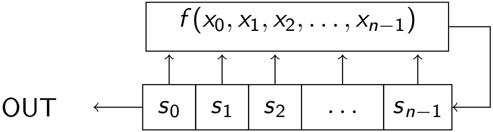

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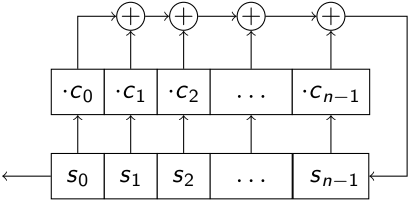

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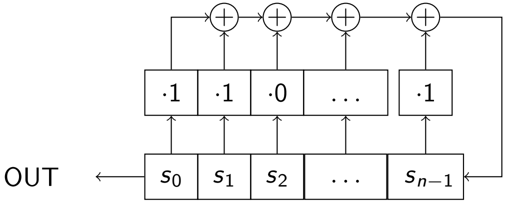

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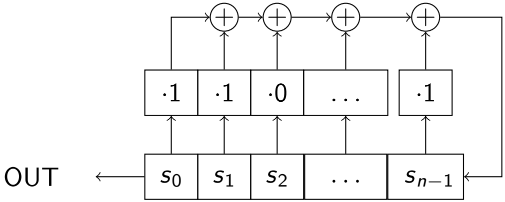

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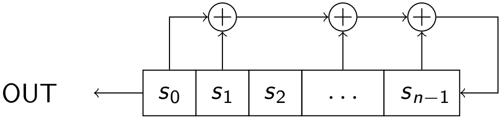

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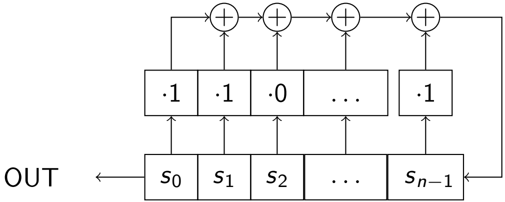

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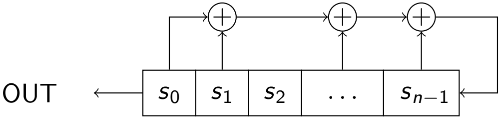

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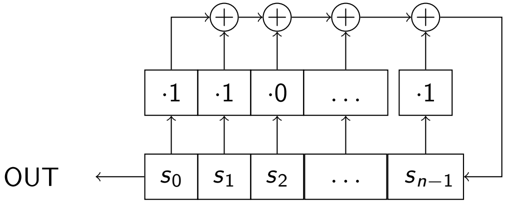

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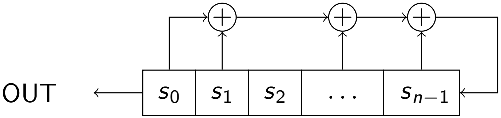

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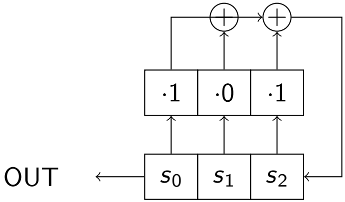

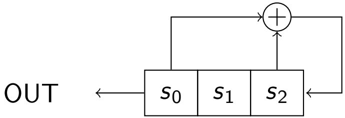

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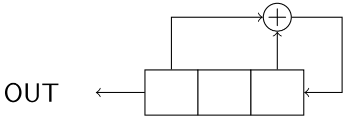

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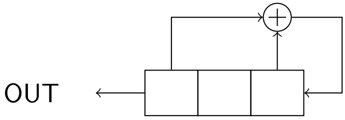

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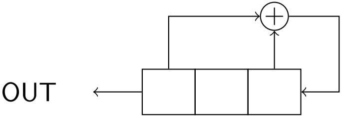

	0	0	1	1
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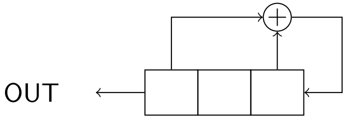

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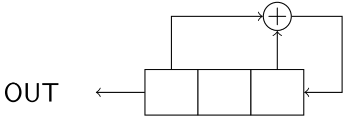

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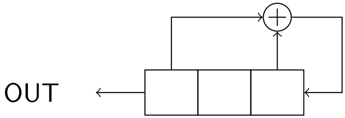

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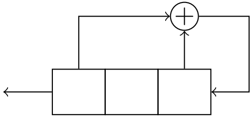

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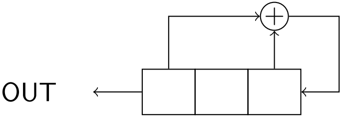

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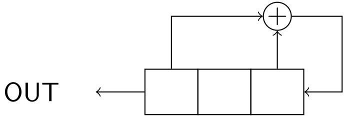
has period 7 with output $\overline{0\,0\,1\,1\,1\,0\,1}$.

This covers all non-zero starting states.



	0	0	1	1
0	0	1	1	1
0	1	1	1	0
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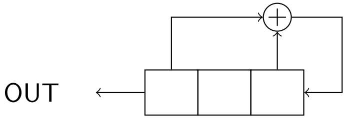

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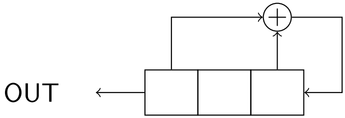
This means that period 7 is maximal
for a register of length 3, as $2^3 - 1 = 7$.



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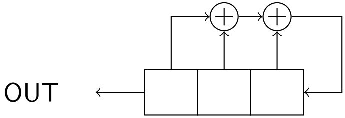

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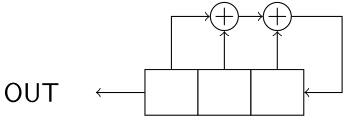

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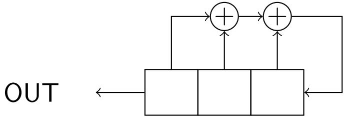

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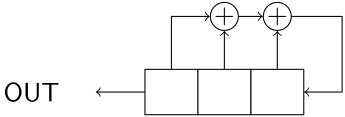

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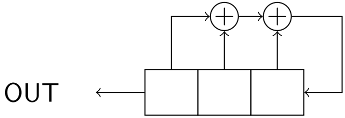

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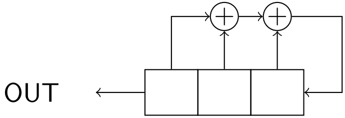
Starting state $S_0 = (0\ 0\ 1)$ gives period 4 with output $\overline{0\,0\,1\,1}$.

This misses $2^3 - 4 = 4$ states.



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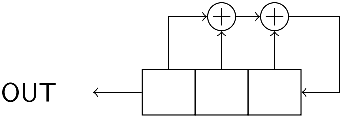

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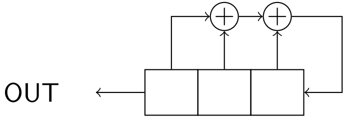

	0	0	1	1
0	0	1	1	0
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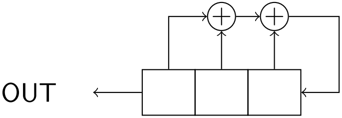

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	0	0	1	1
0	0	1	1	0
0	1	1	0	0
1	1	0	0	1
1	0	0	1	

	1	1	1	1
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	1	0	1	0

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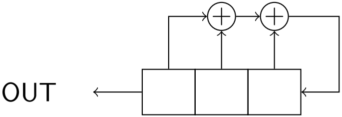
Starting state 1 0 1 gives period 2 with output $\overline{1\,0}$



	0	0	1	1
0	0	1	1	0
0	1	1	0	0
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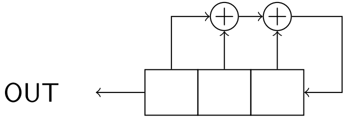

Starting state $S_0 = (0\ 0\ 1)$
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Starting state 1 1 1
gives period 1 with output $\bar{1}$

Starting state 1 0 1
gives period 2 with output $\overline{1\,0}$

Together with $\bar{0}$
we have now seen all 8 states.

Periods are 4,2,1,1
depending on starting state.

	0	0	1	1
0	0	1	1	0
0	1	1	0	0
1	1	0	0	1
1	0	0	1	

	1	1	1	1
1	1	1	1	

	1	0	1	0
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- An attacker knows the size of the register – there are n bits in the IV.
- The output bits have linear relations

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  - *is unpredictable given any previous portion of the stream;*
  - *does not exhibit any non-random statistical properties.*
- We can analyze LFSRs mathematically.
- LFSRs are used in combination with non-linear functions in stream cipher design.