Explicit Complex Multiplication

Benjamin Smith

INRIA Saclay–Île-de-France
& Laboratoire d'Informatique de l'École polytechnique (LIX)

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So, where were we?

In the last lecture, we saw that if E is an elliptic curve and $\mathrm{End}(E)$ is its endomorphism ring, then

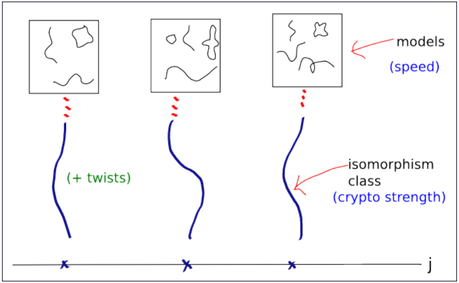
- $\mathrm{End}(E)$ contains the multiplication-by-m map for every m in $\mathbb{Z}$;
- over $\mathbb{F}_q$, we also have the Frobenius endomorphism;
- we also have $\mathrm{Aut}(E) \subset \mathrm{End}(E)$ (but generically $\mathrm{Aut}(E) = \{[\pm 1]\}$, so this doesn't give anything new.)

In this lecture, we want to explore the structure of $\mathrm{End}(E)$.

We use $\mathrm{End}(E)$ to denote the ring of endomorphisms of E defined over k, while $\mathrm{End}_{\overline{k}}(E)$ denotes the endomorphisms of E defined over $\overline{k}$.

More on the j-invariant

First, let's talk a bit more about the j-invariant...



The idea is that there is essentially only one degree of freedom when choosing an elliptic curve over $\mathbb{F}_q$. Choosing a j-invariant and a twist determines your curve and your security.

Choosing the model of your curve makes a difference to your speed, but not your essential cryptographic efficiency.

The structure of $\mathrm{End}(E)$

There are only three kinds of rings that $\mathrm{End}(E)$ can be isomorphic to.

Theorem

Let E be an elliptic curve over k. One of the following holds:

1. $\mathrm{End}(E) = \mathrm{End}_{\overline{k}}(E) \cong \mathbb{Z}$.
2. $\mathrm{End}_{\overline{k}}(E) \cong$ *an order in a quadratic imaginary extension of* $\mathbb{Q}$.
3. $\mathrm{End}_{\overline{k}}(E) \cong$ *an order in a quaternion algebra over* $\mathbb{Q}$.

- If $\mathrm{char}\ k = 0$, then (3) cannot occur (for slightly tricky reasons).
- If $\mathrm{char}\ k \neq 0$, then (1) cannot occur (because π_E is not an integer). Further, (3) occurs if and only if E is supersingular.

If $\mathrm{End}(E) \neq \mathbb{Z}$, then we say that E has **complex multiplication** (**CM**). You should recognise $\mathbb{Z}$, but what about the other rings?

Orders in quadratic imaginary fields

Suppose $K = \mathbb{Q}(\alpha)$ is a quadratic imaginary field
(so α satisfies a quadratic minimal polynomial with negative discriminant.)

The **ring of integers** (or *maximal order*) of K is

$$\mathcal{O}_K = \{\beta \in K : m(\beta) = 0 \text{ for some monic integer polynomial } m\}.$$

The **orders** of K are the subrings $\mathcal{O}$ of K satisfying

- $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-module, and
- $\mathcal{O} \otimes \mathbb{Q} = K$ (that is, K is like $\mathcal{O}$ "with (rational) denominators").

These orders are precisely the subrings of K of the form

$$\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K \quad \text{where } f^2 \text{ divides } \Delta_K \text{ (the discriminant of } K).$$

Orders in quadratic imaginary fields

Example

If $K = \mathbb{Q}(\sqrt{-3})$, then $(1+\sqrt{-3})/2$ has minimal polynomial $X^2 - X + 1$, so $(1+\sqrt{-3})/2$ is in $\mathcal{O}_K$.
In fact $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{-3})/2]$, and $\Delta_K = 12 = 2^2 \cdot 3$.
The orders of K are therefore

$$\mathbb{Z} + 1 \cdot \mathcal{O}_K = \mathcal{O}_K \quad \text{and} \quad \mathbb{Z} + 2 \cdot \mathcal{O}_K = \mathbb{Z}[\sqrt{-3}].$$

Note that $\mathbb{Z}[\sqrt{-3}]$ has index 2 in $\mathcal{O}_K$.

Orders in quaternion algebras

A **quaternion algebra** is an algebra of the form

$$K = \mathbb{Q} + \mathbb{Q}\alpha + \mathbb{Q}\beta + \mathbb{Q}\alpha\beta$$

where α^2 and β^2 are negative rational numbers, and $\alpha\beta = -\beta\alpha$.

An order $\mathcal{O}$ of K is a subring of K such that

- $\mathcal{O}$ is finitely generated as a $\mathbb{Z}$-module, and
- $\mathcal{O} \otimes \mathbb{Q} = K$ (that is, K is like $\mathcal{O}$ "with denominators").

We won't be needing these today, since we will be concentrating on ordinary curves.

Frobenius

Let E be an elliptic curve over $\mathbb{F}_q$, with Frobenius endomorphism π_E.
Recall that π_E has a characteristic polynomial

$$\chi_E(X) = X^2 - t_E X + q \quad \text{with} \quad |t_E| \leq 2\sqrt{q}$$

such that $\chi_E(\pi_E) = 0$.
The discriminant of χ_E is $\Delta = t_E^2 - 4q < 0$,
so $\mathbb{Q}(\pi_E) \cong \mathbb{Q}[X]/(\chi_E(X))$ is a quadratic imaginary field,
and $\mathrm{End}(E)$ is an order in $\mathbb{Q}(\pi_E)$.

We have

$$\mathbb{Z}[\pi_E] \subset \mathrm{End}(E) \subset \mathrm{End}_{\overline{k}}(E) \subset \mathcal{O}_{\mathbb{Q}(\pi_E)}.$$

Remark

Determining $\mathrm{End}(E)$ (and $\mathrm{End}_{\overline{k}}(E)$) is a nontrivial matter,
which is addressed by Kohel's algorithm.

Isogenies and endomorphism rings

Suppose $\phi : E \to F$ is an isogeny. How are $\mathrm{End}(E)$ and $\mathrm{End}(F)$ related?

Definition

If E is an elliptic curve, then we define $\mathrm{End}^0(E) := \mathrm{End}(E) \otimes \mathbb{Q}$.
We call $\mathrm{End}^0(E)$ the **endomorphism algebra** of E.

For each ψ in $\mathrm{End}(F)$, we have an endomorphism $\phi^\dagger \psi \phi$ of E.

Exercise

Show that the map

$$\psi \longmapsto \frac{1}{\deg(\phi)} \phi^\dagger \psi \phi$$

defines an isomorphism $\mathrm{End}^0(F) \to \mathrm{End}^0(E)$.

Theorem

$\mathrm{End}^0(E)$ *is an isogeny class invariant.*

Isogenies and endomorphism rings

Corollary

If $k = \mathbb{F}_q$, then $\mathbb{Q}(\pi_E) \cong \mathbb{Q}(\pi_F)$.

Corollary

The set of supersingular elliptic curves over $\mathbb{F}_p$ is an isogeny class.

If $\phi : E \to F$ is an isogeny, then $\mathrm{End}^0(E) \cong \mathrm{End}^0(F)$,
but we can still have $\mathrm{End}(E) \ncong \mathrm{End}(F)$.
In particular, $\mathrm{End}(E)$ and $\mathrm{End}(F)$ can be different orders in $\mathrm{End}^0(E)$.

However, if ϕ is an l-isogeny (that is, it has degree l), then either

- $\mathrm{End}(E) = \mathrm{End}(F)$, or
- $\mathrm{End}(E)$ has index l in $\mathrm{End}(F)$, or
- $\mathrm{End}(F)$ has index l in $\mathrm{End}(E)$.

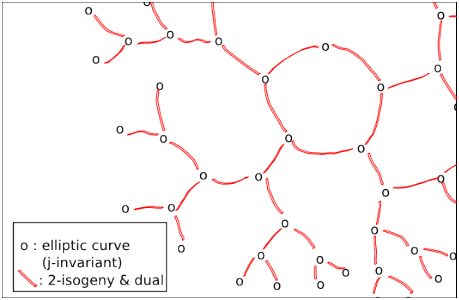
So an isogeny ϕ can change the size of the endomorphism,
but only by an index depending on the degree of ϕ.

A (very) brief look at Kohel's algorithm

Suppose we want to determine $\mathrm{End}(E)$ for some ordiary E over $\mathbb{F}_q$.
First, we compute $t_E = (q+1) - \#E(\mathbb{F}_q)$; then $\chi_E = X^2 - t_E X + q$, so

$$\mathrm{End}(E) = \mathbb{Z} + f \cdot \mathcal{O}_{\mathbb{Q}(\pi_E)}$$

for some f dividing the conductor m of $\mathbb{Z}[\pi_E]$ in $\mathcal{O}_{\mathbb{Q}(\pi_E)}$.
Next, we factor m (which is likely to be smooth, hence easy to factor).
For each prime l dividing m, we construct the l-isogeny graph containing $j(E)$ in the moduli space, which looks something like this:



Kohel's algorithm (continued)

The idea is that the j-invariants in the cycle correspond to curves F with endomorphism ring $\mathrm{End}(F) \cong \mathcal{O}_{\mathbb{Q}(\pi_E)}$, while each step away from the cycle reduces the endomorphism ring by an index l.

The largest power of l dividing f is the distance from $j(E)$ to the cycle.

Morain and Fouquet use these ideas in reverse to speed up the Schoof point counting algorithm.

CM in characteristic zero

What is the situation for elliptic curves over $\mathbb{Q}$?

If E is an elliptic curve over $\mathbb{Q}$ (or $\mathbb{C}$ for that matter), then either

- $\mathrm{End}(E) = \mathbb{Z}$ (the generic situation), or
- $\mathrm{End}(E) \cong$ an order in a quadratic imaginary field (the exceptional case).

Remark

Over $\mathbb{C}$, elliptic curves are isomorphic to complex tori: that is, each curve is a quotient of $\mathbb{C}$ by a lattice $\Lambda = \langle 1, \tau \rangle$. The endomorphisms of $\mathbb{C}/\Lambda$ are the elements $z \in \mathbb{C}$ such that $z\Lambda = \Lambda$. Noninteger endomorphisms can only exist if τ is an algebraic integer, and in fact all of these endomorphisms must lie in $\mathbb{Q}(\tau)$.

Reduction of curves and endomorphisms

Recall that if $E : y^2 = f(x)$ is an elliptic curve defined over $\mathbb{Q}$ and p is a prime of good reduction for E, then reducing the equation of E modulo p defines an elliptic curve $\overline{E} : y^2 = \overline{f}(x)$ over $\mathbb{F}_p$.

If ϕ is an endomorphism of E, then we can reduce the coefficients of its rational map modulo p to give an endomorphism $\overline{\phi}$ of $\overline{E}$.

Theorem

The map $\mathrm{End}(E) \to \mathrm{End}(\overline{E})$ induced by reducing modulo p is an injective homomorphism.

Many curves over $\mathbb{Q}$ reduce to the same $\overline{E}$ modulo p, and $\mathrm{End}(\overline{E})$ "contains" the endomorphism ring of *every one* of them.

The endomorphism algebra of a reduction

Corollary

Let E be an elliptic curve over $\mathbb{Q}$ such that $\mathrm{End}(E)$ is an order $\mathcal{O}$ in a quadratic imaginary field K, and let p be any prime of good reduction for E. Then $\mathrm{End}(\overline{E})$ contains a subring isomorphic to $\mathcal{O}$.

Note that $\mathrm{End}^0(E)$ need not be isomorphic to K.
If $\overline{E}$ is ordinary then $\mathrm{End}^0(E) \cong K$,
but if $\overline{E}$ is supersingular then K is only the center of $\mathrm{End}^0(E)$.

The CM method for curve construction

One application of this result is the **CM method**
(of which we will only give a very rough sketch).

Suppose we have an algorithm that, given a quadratic imaginary field K, together with an element α of K of norm p, constructs an elliptic curve E over $\mathbb{Q}$ such that $\mathrm{End}^0(E) \cong K$ with α representing π_E.

Suppose now that we want a curve F over $\mathbb{F}_p$ such that $\#F(\mathbb{F}_p) = N$.
We know that $t_E = p + 1 - N$, so $\mathrm{End}^0(F)$ must contain the field
$K = \mathbb{Q}[X]/(X^2 - (p + 1 - N)X + p)$.

Applying the algorithm we compute a curve E over $\mathbb{Q}$ with $\mathrm{End}^0(E) \cong K$.
Then we reduce E modulo p to obtain a curve $F = \overline{E}$ over $\mathbb{F}_p$
with the required number of points.

(More generally, E could be defined over a number field.)

Class polynomials

It remains to determine a way to compute an E over $\mathbb{Q}$.

It is enough to compute the j-invariant of E, since this is enough to reconstruct E (though we may need to check the right twist of $\overline{E}$.)

The conventional solution uses the **class polynomial** of K: that is, a polynomial $H_{\Delta_K}(X)$ whose roots are the j-invariants of curves over $\mathbb{Q}$ whose endomorphism ring is equal to $\mathcal{O}_K$.

These class polynomials can be precomputed relatively easily:
Enge's algorithm runs in essentially linear time in size of Δ_K.
The hard part is finding enough disk space to write down the polynomial.

In Magma, you can compute class polynomials using
`HilbertClassPolynomial(Discriminant(K));`

Examples of class polynomials

Example

Let $K = \mathbb{Q}(i) = \mathbb{Q}[X]/(X^2+1)$. The maximal order of K is $\mathbb{Z}[i]$.
The field K has discriminant -4.
The class polynomial of K is $H_{-4}(X) = X - 1728$;
so only curves with j-invariant 1728 can have endomorphism ring isomorphic to $\mathbb{Z}[i]$.
Recall that these curves are all $\overline{Q}$-isomorphic to $E : y^2 = x^3 + x$.

Example

Some examples of other class polynomials include

$$H_{-20}(X) = X^2 - 1264000X - 681472000$$

$$H_{-52}(X) = X^2 - 6896880000X - 567663552000000$$

$$H_{-31}(X) = X^3 + 39491307X^2 - 58682638134X + 1566028350940383$$

Eigenvalues of endomorphisms

Let E be an elliptic curve over $\mathbb{F}_q$, and suppose $E(\mathbb{F}_q)$ has a subgroup G of large prime order N (so N^2 does not divide $\#E(\mathbb{F}_q)$).
For cryptography, we need to do a lot of scalar multiplication in G.

Suppose we can *efficiently compute* a non-integer endomorphism ϕ of E; let $m_\phi(X)$ be its (quadratic) minimal polynomial.

We have $\phi(G) = G$; but G is cyclic, so its endomorphisms are all integer multiplications; so ϕ acts like an integer eigenvalue on G.
Indeed, ϕ acts like $[\lambda]$ on G, where λ is one of the roots of $m_\phi(X)$ mod N.

Given an integer m, we can write

$$[m] = [a] + [\lambda][b]$$

with a and b about half the size of m.
For any P in G, we can then compute $[m]P$ more efficiently by using

$$[m]P = [a]P + \phi([b]P).$$

This is the basis of Gallant–Lambert–Vanstone (**GLV**) multiplication.

Other fast multiplication techniques

Endomorphisms also appear in other fast multiplication techniques.

Example (Multiplication with Frobenius expansions)

Let E be an elliptic curve defined over $\mathbb{F}_p$,
but viewed as an elliptic curve over $\mathbb{F}_q$ where $q = p^n$.
The p-power Frobenius isogeny is then an endomorphism π_p of E.
Again, $\pi_p(G) = G$, so π_p has an eigenvalue λ on G.
When we want to multiply by an integer m, we can expand m in base λ, writing

$$m = m_0 + m_1\lambda + m_2\lambda^2 + \cdots$$

and then evaluate

$$[m]P = [m_0]P + \pi_p([m_1]P) + \pi_p^2([m_2]P) + \cdots .$$

During ECC you will see Galbraith–Scott multiplication, which also uses a fast explicit endomorphism to speed up scalar multiplication on elliptic curves defined over $\mathbb{F}_{q^2}$.