Efficient Arithmetic on Elliptic and Hyperelliptic Curves

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Tutorial on Elliptic and Hyperelliptic Curve Cryptography
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Main question of this talk:

How can we efficiently compute on elliptic and hyperelliptic curves?
Part I: General Concepts

Part II: Fast arithmetic on elliptic curves (characteristic $> 3$)

Part III: Fast arithmetic on hyperelliptic curves (characteristic 2)
Part I : General concepts
Algorithm 1 (Double-and-Add)

IN: An element $P \in G$ and a positive integer $n = (n_{l-1} \ldots n_0)$ where $n_{l-1} = 1$.

OUT: The element $[n]P \in G$.

1. $R \leftarrow P$
2. for $i = l - 2$ to 0 do
   2. if $n_i = 1$ then $R \leftarrow R \oplus P$
   3. $i \leftarrow i - 1$
3. return $R$
Double-and-Add (2)

The algorithm uses the following rule:

\[ [(n_{l-1} \ldots n_i)_2]P = [2]([[n_{l-1} \ldots n_{i+1}]_2]P) \oplus [n_i]P \]

**Example:** \( 45 = (101101)_2 \), compute \([45]P\)

\[
\begin{align*}
P \\
[2]P \\
[2]([2]P) \oplus P \\
[2]([2]([2]P) \oplus P) \oplus P \\
[2]([2]([2]([2]P) \oplus P) \oplus P) \oplus P \\
[2]([2]([2]([2]([2]P) \oplus P) \oplus P)) \oplus P \\
&= [45]P
\end{align*}
\]
Remarks

- $w(n)$ denotes the Hamming weight of $n$. This is the number of nonzero digits in the binary representation of $n$.
- Double-and-Add needs $\ell - 1$ doublings and $w(n)$ additions $\Rightarrow$ doubling is more important than addition!
- **Variant:** Use NAF instead of binary representation (Double-and-Add/Subtract). This is a signed bit representation using the digits $1, 0, -1$.
- In NAF no two consecutive digits are non-zero. The number of additions/subtractions is less than in binary form. We need to compute $-P$ efficiently.
The $2^k$-ary Method

- Use a larger basis to get sparse representations of $n$
- A common choice is $2^k$ as basis
- $S = \{0, 1, \ldots, 2^k - 1\}$ are the digits
- To perform scalar multiplication, first precompute $[s]P$ for all $s \in S$ and use a modified version of Algorithm 1

**Example** $k = 3, S = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$n = 241 = (11|110|001)_2 = (361)_{2^3}$
Algorithm 2 (Left–to–right $2^k$-ary)

IN: An element $P \in G$ and a positive integer $n$ in $2^k$-ary representation $n = (n_{l-1} \ldots n_0)_{2^k}$

OUT: The element $[n]P \in G$.

1. $R \leftarrow [n_{l-1}]P$
2. for $i = l - 2$ to 0 do
   1. $R \leftarrow [2^k]R$
   2. if $n_i \neq 0$ then $R \leftarrow R \oplus [n_i]P$
   3. $i \leftarrow i - 1$
3. return $R$
Windowing Methods (3)

**Example** $k = 3, S = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$n = 241 = (361)_{2^3}$


$R = [2^3]R = [240]P$ (Multiply by $2^3$ and add $P$)

$R = R \oplus [1]P = [241]P$
To reduce the number of precomputations, a **sliding window method** can be used!

- Digits are only the **odd** integers smaller than $2^k$ and 0
- $S = \{0, 1, 3, 5, \ldots, 2^k - 1\}$
- Consecutive zeros are skipped
- Scan from right to left $\Rightarrow$ block is odd

**Example** ($k = 3$)

$$241 = (1 111 000 1)_2$$

- Sliding window is also possible with signed digits!
Part II: Fast arithmetic on elliptic curves
(characteristic $> 3$)
An elliptic curve over $\mathbb{F}_p$ can be given by an equation of the form

$$E : y^2 = x^3 + ax + b,$$

where $a, b \in \mathbb{F}_p$ and the discriminant $4a^3 + 27b^2 \neq 0$.

The points on the curve $E$ are the pairs $(x, y)$ which satisfy the curve equation. The set of $\mathbb{F}_p$-rational points (i.e. points with coeffs. in $\mathbb{F}_p$) form an additive group, denoted by $E(\mathbb{F}_p)$. 
The most important operation (for cryptography) on $E(\mathbb{F}_p)$ is scalar multiplication $[n]P = P + \ldots + P$ for an integer $n$ and a point $P \in E(\mathbb{F}_p)$.

Use for instance double-and-add or windowing methods to compute $[n]P$.

We need fast addition and doubling functions on $E(\mathbb{F}_p)$!

(Doubling is the most important function in double-and-add algorithms.)
An elliptic curve can be represented using several coordinate systems. For each such system, the speed of additions and doublings is different.

We consider affine, projective and Jacobian coordinates and give the appropriate formulas and the number of operations.
Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on $E$.

(Addition) $R = P \oplus Q$ can be computed using the following formulas, where $x_1 \neq x_2$:

$$x_3 = \lambda^2 - x_1 - x_2, \quad y_3 = \lambda(x_1 - x_3) - y_1,$$

where $\lambda = (y_2 - y_1)/(x_2 - x_1)$. **Complexity:** $1I+2M+1S$

(Doubling) $[2]P = P \oplus P$ can be computed using the formulas:

$$x_3 = \lambda^2 - 2x_1, \quad y_3 = \lambda(x_1 - x_3) - y_1$$

where $\lambda = (3x_1^2 + a)/(2y_1)$. **Complexity:** $1I+2M+2S$
Let \( x = X/Z \) and \( y = Y/Z \). \( P = (X, Y, Z) \). We consider the homogenised curve equation

\[
E_P : Y^2Z = X^3 + aXZ^2 + bZ^3.
\]

Let \( P = (X_1, Y_1, Z_1) \) and \( Q = (X_2, Y_2, Z_2) \) be on \( E_P \) and \( R = P \oplus Q \)

**Addition:** \( R = P \oplus Q = (X_3, Y_3, Z_3) \)

\[
X_3 = vA, \quad Y_3 = u(v^2X_1Z_2 - A), \quad Z_3 = v^3Z_1Z_2,
\]

where \( u = Y_2Z_1 - Y_1Z_2 \), \( v = X_2Z_1 - X_1Z_2 \) and \( A = u^2Z_1Z_2 - v^3 - 2v^2X_1Z_2 \)

**Complexity:** \( 12M + 2S \)
Let $P = (X_1, Y_1, Z_1)$.

**Doubling:** \([2] P = P \oplus P = (X_3, Y_3, Z_3)\)

\[
X_3 = hs, \quad Y_3 = w(B - h) - 2RR, \quad Z_3 = sss,
\]

where $XX = X_1^2$, $ZZ = Z_1^2$, $w = aZZ + 3XX$, $s = 2Y_1Z_1$, $ss = s^2$, $sss = s \cdot ss$, $R = Y_1s$, $RR = R^2$, $B = (X_1 + R)^2 - XX - RR$, $h = w^2 - 2B$

**Complexity:** 6M+6S (see Explicit-Formulas Database, http://www.hyperelliptic.org/EFD)
In Jacobian coordinates, we set $x = X/Z^2$ and $y = Y/Z^3$. The weighted homogenised curve equation is:

$$E_J : Y^2 = X^3 + aXZ^4 + bZ^6.$$ 

Let $P = (X_1, Y_1, Z_1)$ and $Q = (X_2, Y_2, Z_2)$ be on $E_J$.

The addition $P \oplus Q = (X_3, Y_3, Z_3)$ works as follows:

$$X_3 = r^2 - J - 2V, \quad Y_3 = r(V - X_3) - 2S_1J, \quad Z_3 = ((Z_1 + Z_2)^2 - ZZ_1 - ZZ_2)H,$$

where $ZZ_1 = Z_1^2$, $ZZ_2 = Z_2^2$, $U_1 = X_1ZZ_2$, $U_2 = X_2ZZ_1$, $S_1 = Y_1Z_2ZZ_2$, $S_2 = Y_2Z_1ZZ_1$, $H = U_2 - U_1$, $I = (2H)^2$, $J = HI$, $r = 2(S_2 - S_1)$, $V = U_1I$

The complexity of the addition is: $11M + 5S$
Let \( P = (X_1, Y_1, Z_1) \) be on \( E_J \).

The **doubling** \([2]\) \( P = P \oplus P = (X_3, Y_3, Z_3) \) works as follows:

\[
X_3 = T,
Y_3 = M(S - T) - 8YYYY,
Z_3 = (Y_1 + Z_1)^2 - YY - ZZ,
\]

where \( XX = X_1^2 \), \( YY = Y_1^2 \), \( YYYY = YY^2 \), \( ZZ = Z_1^2 \),
\[
S = 2((X_1 + YY)^2 - XX - YYYY),
M = 3XX + aZZ^2,
T = M^2 - 2S
\]

The **complexity** of the doubling is: \( 2M + 8S \)
Addition and doubling on elliptic curves using different coordinate systems (characteristic $p > 3$)

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>Addition</th>
<th>Doubling</th>
<th>Doubling*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Affine</td>
<td>$1I+2M+1S$</td>
<td>$1I+2M+2S$</td>
<td>$14,6M$</td>
</tr>
<tr>
<td>Projective</td>
<td>$12M+2S$</td>
<td>$6M+6S$</td>
<td>$10,8M$</td>
</tr>
<tr>
<td>Jacobian</td>
<td>$11M+5S$</td>
<td>$2M+8S$</td>
<td>$8,4M$</td>
</tr>
</tbody>
</table>

(Doubling*: As an example we assume $I/M = 11$ and $S/M = 0.8$.)
Part III : Fast arithmetic on hyperelliptic curves (characteristic 2)
A hyperelliptic curve $C$ (of genus 2) over $\mathbb{F}_{2^d}$ can be given by an equation of the form

$$C : y^2 + h(x)y = f(x),$$

where $h \in \mathbb{F}_{2^d}[X]$ is of degree at most 2 and $f \in \mathbb{F}_{2^d}[X]$ is monic of degree 5.

The points on the curve $C$ do not form a group! We use the divisor class group of $C$ instead. (Recall, that a divisor is a formal sum of points.)

**Goal:** We want to have fast doubling in the divisor class group!
How to represent elements of the divisor class group?

**Theorem (Mumford)**

Let $C$ be a hyperelliptic curve of genus $g$ over an arbitrary field $K$. Each nontrivial divisor class of $C$ over $K$ can be represented by a unique pair of polynomials $u, v \in K[x]$, where

1. $u$ is monic,
2. $\deg v < \deg u \leq g$,
3. $u \mid v^2 + vh - f$.

**Example:** $(C : y^2 + xy = x^5 + x^4 + x^2 + x$ over $\mathbb{F}_{27})$

$D = [x^2 + z^{31}x + z^{61}, z^{43}x + z^{90}]$ is a proper divisor class of $C$, where $z$ is a generator of $\mathbb{F}_{27}^*$. 
Algorithm 14.7 Cantor’s algorithm

INPUT: Two divisor classes $\overline{D}_1 = [u_1, v_1]$ and $\overline{D}_2 = [u_2, v_2]$ on the curve $C : y^2 + h(x)y = f(x)$.

OUTPUT: The unique reduced divisor $D$ such that $\overline{D} = \overline{D}_1 \oplus \overline{D}_2$.

1. $d_1 \leftarrow \gcd(u_1, u_2)$ \[d_1 = e_1u_1 + e_2u_2\]
2. $d \leftarrow \gcd(d_1, v_1 + v_2 + h)$ \[d = c_1d_1 + c_2(v_1 + v_2 + h)\]
3. $s_1 \leftarrow c_1e_1$, $s_2 \leftarrow c_1e_2$ and $s_3 \leftarrow c_2$
4. $u \leftarrow \frac{u_1u_2}{d^2}$ and $v \leftarrow \frac{s_1u_1v_1 + s_2u_2v_1 + s_3(v_1v_2 + f)}{d} \mod u$
5. repeat
6. $u' \leftarrow \frac{f - vh - v^2}{u}$ and $v' \leftarrow (-h - v) \mod u'$
7. $u \leftarrow u'$ and $v \leftarrow v'$
8. until $\deg u \leq g$
9. make $u$ monic
10. return $[u, v]$

(in: Handbook of Elliptic and Hyperelliptic Curve Cryptography, p. 308)
Remarks:

1. Cantor’s algorithm is completely general and holds for all genera, all fields and all (valid) inputs.

2. From this algorithm all the explicit formulas are derived.

3. To get efficient formulas, we use Cantor but restrict to very special cases, e.g. restricting to genus 2, even characteristic and taking $\overline{D_1} = \overline{D_2}$ leads to efficient doubling formulae.

4. The efficiency of the formulas heavily depends on the degree of the polynomial $h$ in the curve equation (see later).
### Doubling, deg \( u = 2 \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Expression</th>
<th>odd</th>
<th>even</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( v \equiv (h + 2v) \mod u = v_1 x + v_0: ) ( v_1 = h_1 + 2v_1 - h_2 u_1, v_0 = h_0 + 2v_0 - h_2 u_0: )</td>
<td>2S, 3M</td>
<td>2S, 3M</td>
</tr>
<tr>
<td>2</td>
<td>compute resultant ( r = \text{res}(\nu, u): ) ( w_0 = \nu_1^2, w_1 = u_1^2, w_2 = \nu_1^2, w_3 = u_1 v_1, r = u_0 w_2 + \nu_0 (\nu_1 - w_3): )</td>
<td>( (w_2 = 4w_0) ) (see below)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>compute almost inverse ( \text{inv}^r = \text{invr}: ) ( \text{inv}^r_1 = -v_1, \text{inv}^r_0 = v_0 - w_3: )</td>
<td>1M</td>
<td>2M (see below)</td>
</tr>
<tr>
<td>4</td>
<td>compute ( k' = (f - hh - vv^2) / u \mod u = k'_1 x + k'_0: ) ( w_3 = f_3 + w_1, w_4 = 2u_0, k'_1 = 2(w_1 - f_4 u_1) + w_3 - w_4 - h_2 v_1; ) ( k'_0 = u_1 (2w_4 - w_3 + f_4 u_1 + h_2 v_1) + f_2 - w_0 - 2f_4 u_0 - h_1 v_1 - h_2 v_0; )</td>
<td>5M</td>
<td>5M</td>
</tr>
<tr>
<td>5</td>
<td>compute ( s' = k' \text{inv}^r \mod u: ) ( w_0 = k'_0 \text{invr}_0, w_1 = k'_1 \text{invr}_1, s'_1 = (\text{invr}_0 + \text{invr}_1) (k'_0 + k'_1) - w_0 - w_1 (1 + u_1), s'_0 = w_0 - u_0 w_1; )</td>
<td>I, 2S, 5M</td>
<td>I, 2S, 5M</td>
</tr>
<tr>
<td>6</td>
<td>compute ( s'' = x + s_0 / s_1 ) and ( s_1: ) ( w_1 = 1 / (r s'_1) (= 1 / r^2 s_1), w_2 = r w_1 (= 1 / s'_1), w_3 = s'_1 w_1 (= s_1); ) ( w_4 = r w_2 (= 1 / s'_1), w_5 = w_4^2, s'_0 = v_0 w_2; )</td>
<td>2M</td>
<td>2M</td>
</tr>
<tr>
<td>7</td>
<td>compute ( l' = s'' u = x^3 + l'_1 x + l'_0: ) ( l'_2 = u_1 + s'_0, l'_1 = u_1 s'_0 + u_0, l'_0 = u_0 s'_0; )</td>
<td>S, 2M</td>
<td>S, M</td>
</tr>
<tr>
<td>8</td>
<td>compute ( u' = s'' + (h + 2v) s / u + (v^2 + h - f) / u^2: ) ( u'_0 = s''_0^2 + w_4 (s''_0 - u_1) + 2v_1 + h_1 + w_5 (2u_1 - f_4), u'_1 = 2s''_0 + h_2 w_4 - w_5; )</td>
<td>4M</td>
<td>4M</td>
</tr>
<tr>
<td>9</td>
<td>compute ( v' \equiv -h - (l + v) \mod u' = v'_1 x + v'_0: ) ( w_1 = l'_2 - u'_1, w_2 = u'_1 w_1 + u'_0 - l'_1, v'_1 = w_2 v_3 - v_1 - h_1 + h_2 u'_1; ) ( w_2 = u'_0 w_1 - l'_0, v'_0 = w_2 v_3 - v_0 - h_0 + h_2 u'_0; )</td>
<td>4M</td>
<td>4M</td>
</tr>
</tbody>
</table>

**total**

| each | 1, 5S, 22M |
Restrict to the case \( \text{deg}(h) = 1 \) (genus 2, characteristic 2)

### Doubling \( \text{deg} h = 1, \text{deg} u = 2 \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Expression</th>
<th>( h_1 = 1 )</th>
<th>( h_1^{-1} ) small</th>
<th>( h_1 ) arbitrary</th>
</tr>
</thead>
</table>
| 1    | compute \( rs_1 \):  
\( z_0 = u_0^2, k'_1 = u_1^2 + f_3; \)  
\( w_0 = f_0 + v_0^2 (= rs'_1/h_1^3); \)  | 3S             | 3S              | 3S              |
| 2    | compute \( 1/s_1 \) and \( s'_0 \):  
\( w_1 = (1/w_0)z_0 (= h_1/s_1); \)  
\( z_1 = k'_1w_1, s''_0 = z_1 + u_1; \) | I, 2M          | I, 2M           | I, 2M           |
| 3    | compute \( u' \):  
\( w_2 = h_1^2w_1, u'_1 = w_2w_1; \)  
\( u'_0 = s''_0^2 + w_2; \) | 2S             | S, 2M           | S, 2M           |
| 4    | compute \( v' \):  
\( w_3 = w_2 + k'_1; \)  
\( v'_1 = h_1^{-1}(w_3z_1 + w_2u'_1 + f_2 + v'_1^2); \)  
\( v'_0 = h_1^{-1}(w_3u'_0 + f_1 + z_0); \) | S, 3M          | S, 3M           | S, 5M           |
| total| I, 6S, 5M  | I, 5S, 7M     | I, 5S, 9M       |
Why do we need inversion-free formulae?

- In hardware applications an inverter might be very expensive (compared to other components).
- I/M-ratio depends on the CPU or platform.

**Example:** IBM Thinkpad X41 with Linux and GMP has an I/M ratio between 10 and 13 for $F_{2d}$, where $7 \leq d \leq 113$.

Asuming $I/M = 13$ and $S/M = 0.8$, the affine doubling forumulae $1I+5M+6S$ correspond to $13M+5M+4.8M=22.8M$.

Can we do better in this setting?
In **projective coordinates** a divisor class is now represented by the quintuple \([U_1, U_0, V_1, V_0, Z]\), corresponding to the divisor class \([x^2 + U_1/Zx + U_0/Z, V_1/Zx + V_0/Z]\) in Mumford representation.

In **recent coordinates** \([U_1, U_0, V_1, V_0, Z, z]\) corresponds to \([x^2 + U_1/Zx + U_0/Z, V_1/Z^2x + V_0/Z^2]\) and \(z = Z^2\).

**New coordinates:** \([U_1, U_0, V_1, V_0, Z_1, Z_2]\) corresponds to \([x^2 + U_1/Z_1^2x + U_0/Z_1^2, V_1/(Z_1^3Z_2)x + V_0/(Z_1^3Z_2)]\).
### Doubling

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>[U_1, U_0, V_1, V_0, Z]</td>
<td>[U'_1, U'_0, V'_1, V'_0, Z'] = [2U_1, U_0, V_1, V_0, Z]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step</th>
<th>Expression</th>
<th>odd</th>
<th>even</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>compute resultant and precomputations: (h_1 = h_1Z; h_0 = h_0Z, Z_2 = Z^2; V_1 = h_1 + 2V_1 - h_2U_1); (V_0 = h_0 + 2V_0 - h_1U_0, w_0 = V_0^2, w_1 = U_1^2, w_2 = V_1^2); (w_3 = V_0Z - U_1V_1, r = V_0w_0 + U_2U_0);</td>
<td>3S, 4M</td>
<td>(w_2 = 4w_0)</td>
</tr>
<tr>
<td>2</td>
<td>compute almost inverse: (i_{\nu V_1} = -V_1, i_{\nu V_0} = w_3);</td>
<td>5M (see below)</td>
<td>5M (see below)</td>
</tr>
<tr>
<td>3</td>
<td>compute (k): (w_3 = f_1Z_2 + w_1, w_4 = 2U_0); (k_1 = 2w_1 + w_4 - Z(w_4 + 2f_1U_1 + h_1V_1)); (k_0 = U_1(Z(2w_4 + f_1U_1 + h_1V_1) - w_3)) + (Z(Z(f_1Z - V_1 - V_0h_1 - V_0h_1 - 2f_0U_0) - w_0))</td>
<td>7M</td>
<td>7M</td>
</tr>
<tr>
<td>4</td>
<td>compute (s = \text{kinv mod } w; w_0 = k_0\nu w_0, w_1 = k_1\nu w_1; s_5 = (i_{\nu V_0} + i_{\nu V_1})(k_0 + k_1) - w_0 - (1 + U_1)w_1) (s_1 = s_3Z, s_0 = w_0 - ZU_0w_1;) If (s_1 = 0) different case</td>
<td>2S, 6M</td>
<td>2S, 6M</td>
</tr>
<tr>
<td>5</td>
<td>precomputations: (R = Zr, R = Rs_1, S_1 = s_1^2, S_0 = s_0^2, t = h_2s_0; s_1 = s_3s_1, s_0 = s_0s_1, S = s_0Z, R = Rs_1;)</td>
<td>3M</td>
<td>3M</td>
</tr>
<tr>
<td>6</td>
<td>compute (l): (l_2 = U_1s_1s_0 = U_0s_0, l_1 = (s_1 + s_0)(U_1 + U_0) - l_2 - l_0;)</td>
<td>1S, 4M</td>
<td>1S, 2M</td>
</tr>
<tr>
<td>7</td>
<td>compute (U'): (U'_0 = s_0 + R(s_0(2V_1 - h_2U_1 + h_1) + t + Zr(2U_1 - l_2Z)); U'_1 = 2S + h_2R - R^2;)</td>
<td>4M</td>
<td>4M</td>
</tr>
<tr>
<td>8</td>
<td>precomputations: (l_2 = l_2 + S - U'_1, w_0 = U'_0l_2 - S_0l_0, w_1 = U'_1l_2 + S_1(U'_0 - l_1);)</td>
<td>3M</td>
<td>3M</td>
</tr>
<tr>
<td>9</td>
<td>adjust: (Z = s_0R, U'_1 = \tilde{R}U'_1, U'_0 = \tilde{R}U'_0;)</td>
<td>2M</td>
<td>2M</td>
</tr>
<tr>
<td>10</td>
<td>compute (V'): (V'_0 = w_0 + h_2U'_0 - \tilde{R}(V_0 - h_0); V'_1 = w_1 + h_2U'_1 - \tilde{R}(V_1 - h_1);)</td>
<td>6S, 38M</td>
<td>7S, ≤38M</td>
</tr>
</tbody>
</table>

(in: Tanja Lange: Formulae for Arithmetic on Genus 2 Hyperelliptic Curves, AAECC 2004)
Inversion-free Doubling in Recent Coordinates

Algorithm 14.51 Doubling in recent coordinates ($g = 2$, $h_2 = 0$, and $q$ even)

INPUT: A divisor class $[U_1, U_0, V_1, V_0, Z, z]$ and the precomputed values $h_1^2$ and $h_1^{-1}$.

OUTPUT: The divisor class $[U'_1, U'_0, V'_1, V'_0, Z', z'] = [2][U_1, U_0, V_1, V_0, Z, z]$.

1. **precomputations**
   
   $Z_4 \leftarrow z^2, y_0 \leftarrow U_0^2, t_1 \leftarrow U_1^2 + f_3z$ and $w_0 \leftarrow Z_4 f_0 + V_0^2$
   
   $\tilde{Z} \leftarrow z w_0, w_1 \leftarrow y_0 Z_4, y_1 \leftarrow t_1 y_0 z$ and $s_0 \leftarrow y_1 + U_1 w_0 Z$
   
   $w_2 \leftarrow h_1^2 w_1$ and $w_3 \leftarrow w_2 + t_1 w_0$

   **[10M + 4S]**

2. **compute $U'$**

   $U'_1 \leftarrow w_2 w_1, w_2 \leftarrow w_2 \tilde{Z}$ and $U'_0 \leftarrow s_0^2 + w_2$

   **[2M + S]**

3. **compute $V'$**

   $Z' \leftarrow \tilde{Z}^2$ and $V'_1 \leftarrow h_1^{-1}(w_2 U'_1 + (w_3 y_1 + f_2 Z' + (V_1 w_0)^2) Z')$
   
   $V'_0 \leftarrow h_1^{-1}(\tilde{Z}(w_3 U'_0 + y_0 w_0 Z'))$, $z' \leftarrow Z'^2$

   **[11M + 3S]**

4. **return** $[U'_1, U'_0, V'_1, V'_0, Z', z']$

   **[total complexity: 23M + 8S]**

(in: Handbook of Elliptic and Hyperelliptic Curve Cryptography, p. 347)

If $h_1 = 1$ one saves 3M. Hence, for $h(x) = x$ a doubling costs 20M+8S.
Doubling on hyperelliptic curves using different coordinate systems in characteristic 2

<table>
<thead>
<tr>
<th>Coordinates</th>
<th>DBL with general $h(x)$</th>
<th>DBL with $h(x) = x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Affine</td>
<td>1I+22M+5S</td>
<td>1I+5M+6S</td>
</tr>
<tr>
<td>Projective</td>
<td>38M+7S</td>
<td>22M+6S</td>
</tr>
<tr>
<td>New</td>
<td>39M+6S</td>
<td>n/a</td>
</tr>
<tr>
<td>Recent</td>
<td>n/a</td>
<td>20M+8S</td>
</tr>
</tbody>
</table>

**Conclusion:** If we restrict to special cases in terms of genus, field (characteristic) and degree of $h$, and choose the right coordinates, we get the best performance for scalar multiplication on HEC.
Thank you for your attention!