Edwards Curves and the ECM Factorisation Method

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Outline

1. What is ECM and how does it work?
2. Edwards curves
3. How can Edwards curves make ECM faster?

Pollard's p-1 Method (1)

Problem: Find a prime factor p of the composite integer N.

- **Fermat's little theorem:** $a^{p-1} \equiv 1 \bmod p$, if p prime and a coprime to p.
- We pick a random element $a \in \{2, \ldots, N-1\}$ and fix a smoothness bound B.
- We hope for $p-1$ (or the order of $a \bmod p$) to be B-powersmooth, i.e. all prime powers $\leq B$.
- Set $R := \mathrm{lcm}(1, \ldots, B)$.
- $\mathrm{ord}(a) \bmod p$ is B-powersmooth $\Rightarrow$ R is a multiple of $\mathrm{ord}(a)$. Thus $a^R \equiv a^{k \cdot \mathrm{ord}(a)} \equiv 1 \bmod p \Rightarrow p \,|\, a^R - 1$.

Result: $\gcd(a^R - 1, N)$ is a factor of N.

Pollard's p-1 Method (2)

This method can fail for two reasons:

1. N does not have a prime divisor p and an element a such that $\mathrm{ord}(a) \bmod p$ is B-powersmooth, i.e. $\gcd(a^R - 1, N) = 1$.

   $\rightarrow$ Increase smoothness bound B.

   $\rightarrow$ Or pick a new a.

2. All prime divisors of N are found simultaneously, i.e. $\gcd(a^R - 1, N) = N$.

   $\rightarrow$ Pick another $1 < a < N$ and try again.

   $\rightarrow$ Ensure that $\mathrm{ord}(a)$ is **not** B-powersmooth modulo all primefactors of N at the same time. Decrease smoothness bound B.

Lenstra's Elliptic Curve Factorisation Method (ECM)

Problem: Find a factor of the composite integer N.

- Let p be a prime factor of N.
- Choose an elliptic curve E over $\mathbb{Q}$ (but reduce $\bmod N$).
- Set $R := \mathrm{lcm}(1, \ldots, B)$ for some smoothness bound B.
- Pick a random point P on E (over $\mathbb{Z}/N\mathbb{Z}$) and compute $Q = [R]P$. In projective coordinates: $Q = (X : Y : Z)$.
- If the order ℓ of P modulo p is B-powersmooth then $\ell \mid R$ and hence Q modulo p is the neutral element $(0 : 1 : 0)$ of E modulo p.

  Thus, the X and Z-coordinates of Q are multiples of p.

  $\Rightarrow\ \gcd(X, N)$ and $\gcd(Z, N)$ are divisors of N.

Remarks

- Big advantage over Pollard p-1: We can vary the curve, which increases the chance of finding at least one curve such that P has smooth order modulo p.

  Using Pollard p-1 we are restricted to $\mathbb{Z}/p\mathbb{Z}$.

- When computing $Q = [R]P$ in affine coordinates, the inversion in $\mathbb{Z}/N\mathbb{Z}$ can fail since $\mathbb{Z}/N\mathbb{Z}$ is not a field. In this case the $\gcd$ of N and the element to be inverted is $\neq 1$.

  $\rightarrow$ Hence we have already found a divisor of N.

- Normally one uses Montgomery curves for ECM. We replace them with Edwards curves since the arithmetic is faster.

Suitable Elliptic Curves for ECM (1)

- For ECM we use elliptic curves over $\mathbb{Q}$ (rank > 0) which have a prescribed torsion subgroup. When reducing those modulo p, we know already some divisors of the group order.

- **Theorem.** Let $E/\mathbb{Q}$ be an elliptic curve and let m be a positive integer such that $\gcd(m,p)=1$. If E modulo p is non-singular the reduction modulo p

$$E(\mathbb{Q})[m] \to E(\mathbb{F}_p)$$

  is injective.

  $\Rightarrow$ The order of the m-torsion subgroup divides $\#E(\mathbb{F}_p)$.

  In particular this increases the smoothness chance of the group order of $E(\mathbb{F}_p)$.

Suitable Elliptic Curves for ECM (2)

Summary

- We want curves with large torsion group over $\mathbb{Q}$.
- We need a generator P of the non-torsion part. Then we can reduce $Q = [R]P$ modulo N for many different values of N (smoothness bound fixed).
- For efficient computation of $Q = [R]P$ we like to have cheap additions. Hence P should have small height.

The Atkin and Morain Construction (1)

- Atkin and Morain give a construction method for elliptic curves over $\mathbb{Q}$ with rank > 0 and torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ and a point with infinite order.
- **Advantage:** Infinite family of curves with large torsion and rank 1.
- **Disadvantage:** Large height of the points and parameters slow down the scalar multiplication.

The Atkin and Morain Construction (2)

Example

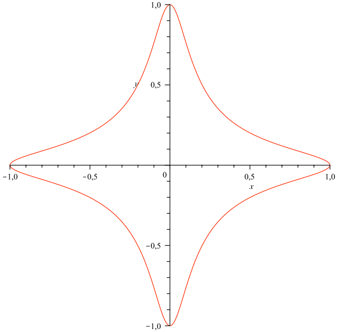
The curve $E : y^2 = x^3 + 212335199041/4662158400x^2 - 202614718501/22106401080x + 187819091161/419284740484$ has torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ and rank 1.

This curve has good reduction at $p = 641$. The group of points on E modulo p is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/336\mathbb{Z}$ and 16 divides $\#E(\mathbb{F}_{641})$ according to the theorem.

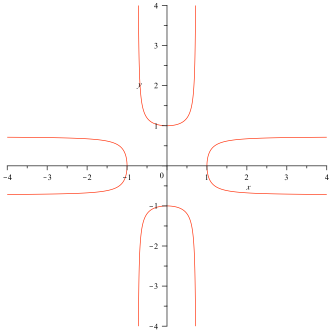
2. Edwards Curves

What is an Edwards curve? (1)

- Let k be a field with $2 \neq 0$ and $d \in k \setminus \{0,1\}$.
- An Edwards curve over k is a curve with equation $x^2 + y^2 = 1 + dx^2y^2$.



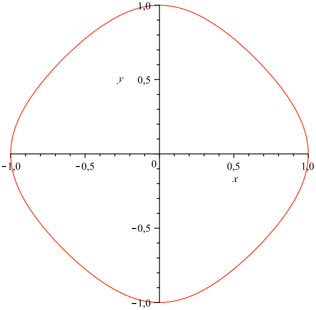
$d = -70$



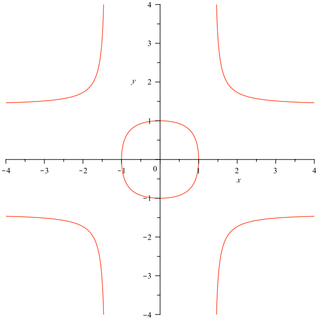
$d = 1.9$

What is an Edwards curve? (2)

- In 2007, Harold M. Edwards introduced a new normal form for elliptic curves.
- Lange and Bernstein slightly generalised this form for use in cryptography, and provided explicit addition and doubling formulas (see Asiacrypt 2007).



$d = -1$



$d = 1/2$

Addition Law on Edwards Curves

Addition on the curve $x^2 + y^2 = 1 + dx^2y^2$

$$(x_1, y_1) + (x_2, y_2) = \left(\frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2}, \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2} \right)$$

Doubling formula (addition with $x_1 = x_2$ and $y_1 = y_2$)

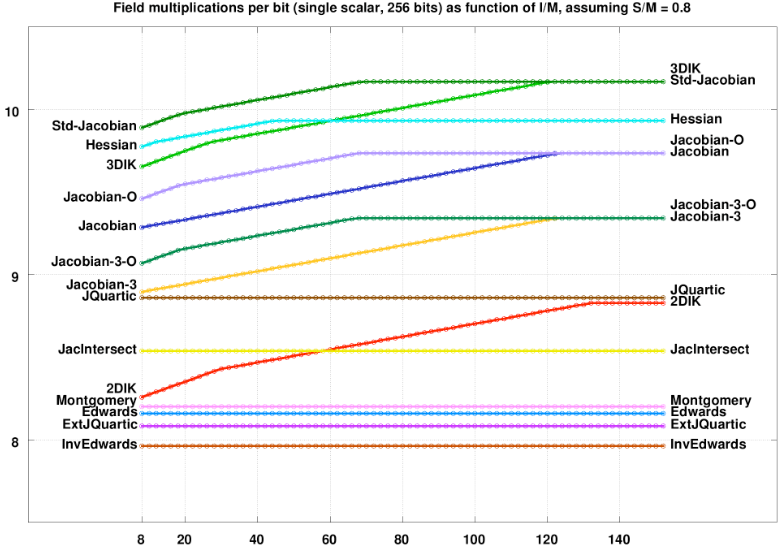
$$[2](x_1, y_1) = \left(\frac{2x_1y_1}{1 + dx_1^2y_1^2}, \frac{y_1^2 - x_1^2}{1 - dx_1^2y_1^2} \right)$$

- The neutral element is $(0, 1)$.
- The negative of a point (x, y) is $(-x, y)$.

The Edwards Addition Law is Complete

- For d not a square in k, the Edwards addition law is complete, i.e. there are no exceptional cases
- Edwards addition law allows omitting all checks
  - Neutral element is affine point on the curve
  - Addition works to add P and P
  - Addition works to add P and $-P$
  - Addition just works to add P and any Q
- Only complete addition law in the literature

Edwards Curves are Fast!



3. How can Edwards curves make ECM faster?

ECM using Edwards Curves (1)

- We can construct Edwards curves over $\mathbb{Q}$ (rank > 0) with prescribed torsion-part and small parameters, and find a point in the non-torsion subgroup.
- To compute $[R]P$ for ECM we use inverted Edwards coordinates which offer very fast scalar multiplication.
- The point in the non-torsion part has small height. This means that all additions in the scalar multiplication are additions with a small point.
- **Example:** $N = (5^{367} + 1)/(2 \cdot 3 \cdot 73219364069)$
  GMP-ECM: 210299 mults. modulo N in 2448 ms.
  GMP-EECM: 195111 mults. modulo N in 2276 ms.
  $\rightarrow$ Speed-up of 7% in first experiments.

ECM using Edwards Curves (2)

- **Theorem of Mazur.** Let $E/\mathbb{Q}$ be an elliptic curve. Then the torsion subgroup $E_{\text{tors}}(\mathbb{Q})$ of E is isomorphic to one of the following fifteen groups:

$$\mathbb{Z}/n\mathbb{Z} \text{ for } n = 1,2,3,4,5,6,7,8,9,10 \text{ or } 12$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \text{ for } n = 1,2,3,4.$$

- All Edwards curves have two points of order 4.
- For ECM we are interested in large torsion subgroups. By Mazur's theorem the largest choices are $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/12\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.
- An Edwards curve over $\mathbb{Q}$ with torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ is not possible. (Also no twisted Edwards curve! See Paper for details.)

Edwards Curves with Torsion Part $\mathbb{Z}/12\mathbb{Z}$

How can we find Edwards curves with prescribed torsion part?

- All Edwards curves have 2 points of order 4, namely $P_4 = (1,0)$ and $P_4' = (-1,0)$.
- We construct a point P_3 of order 3 and obtain a curve with torsion part isomorphic to $\mathbb{Z}/12\mathbb{Z}$ generated by the point $P_{12} = P_3 + P_4$ of order 12.
- We can also ensure that the rank is greater than 0 and determine a point in the non-torsion part which has small height.

Edwards Curves with a Point of Order 3

- Tripling formulas derived from addition law:

  $$[3](x_1,y_1) = \left(\frac{((x_1^2+y_1^2)^2-(2y_1)^2)}{4(x_1^2-1)x_1^2-(x_1^2-y_1^2)^2}x_1, \frac{((x_1^2+y_1^2)^2-(2x_1)^2)}{-4(y_1^2-1)y_1^2+(x_1^2-y_1^2)^2}y_1\right)$$

- For a point P_3 of order 3 we have $[3]P = (0,1)$. (Note, that for a point of order 6 we have $[3]P = (0,-1)$.)
- Thus, the condition is: $\frac{((x_1^2+y_1^2)^2-(2x_1)^2)}{-4(y_1^2-1)y_1^2+(x_1^2-y_1^2)^2}y_1 = \pm 1$
- **Theorem.** If $u \in \mathbb{Q} \setminus \{0, \pm 1\}$ and

  $$x_3 = \frac{u^2-1}{u^2+1},\ y_3 = \frac{(u-1)^2}{u^2+1},\ d = \frac{(u^2+1)^3(u^2-4u+1)}{(u-1)^6(u+1)^2},$$

  then (x_3,y_3) is a point of order 3 on the Edwards curve given by $x^2+y^2 = 1+dx^2y^2$.

Edwards Curves with Torsion Part $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$

- If d is a rational square, then we have 2 more points of order 2 on the Edwards curve. If we additionally enforce that the curve has a point of order 8, the torsion group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ (due to Mazur).

- We always have 2 points of order 4, namely $(\pm 1, 0)$. For a point P_8 of order 8 we need $[2]P_8 = (\pm 1, 0)$.
  $\rightarrow$ Solve this equation using the doubling formulas.

- We get a parametrisation for this solution: If $u \neq 0, -1, -2$, then $x_8 = (u^2 + 2u + 2)/(u^2 - 2)$ gives $P_8 = (x_8, x_8)$, which has order 8 on the curve given by $d = (2x_8^2 - 1)/x_8^4$.

How to Find Curves with Rank 1?

- Until now we have constructed Edwards curves over $\mathbb{Q}$ with torsion subgroup $\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.
- Which of them have rank > 0?
- For both cases we have a parametrisation: A rational number u gives a curve with the desired torsion subgroup.
- To find a curve with rank 1, put $u = a/b$ and do a exhaustive search for solutions (a, b, e, f), where (e, f) is a point on the curve but different from all torsion points, i.e. different from $\{(0, \pm 1), (\pm 1, 0)\}$ etc. Points of order 8 can be excluded by checking for $e = f$.

  Then the point (e, f) has infinite order over $\mathbb{Q}$.

Advantages of GMP-EECM over GMP-ECM (1)

- We choose curves with large torsion subgroups (12 or 16 points) and therefore large guaranteed divisors of the order of $\#E$ modulo p. GMP-ECM uses Suyama curves which have a rational torsion group of order 6.
- We choose curves with parameters and non-torsion points of small height (smaller than Atkin-Morain) and our implementation takes this into account by working with projective base points and projective parameters. The GMP-ECM implementation does not make use of small height elements and instead computes every fraction a/b modulo p which means that the numbers get big.

Advantages of GMP-EECM over GMP-ECM (2)

- In inverted Edwards coordinates the cost of a scalar multiplication is $1DBL + \varepsilon ADD$ per bit, where $\varepsilon \to 0$ when the scalar gets large, i.e. asymptotically $3M + 4S + 1D$.

  GMP-ECM uses Montgomery curves. The Montgomery ladder needs $5M + 4S + 1D$ per bit; GMP-ECM uses the PRAC algorithm instead of the latter. It needs an average of $9M$ per bit.

Summary

Until now we already have

- 100 curves with small parameters and torsion subgroup $\mathbb{Z}/12\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.
- Complete translation of the Atkin-Morain method to Edwards curves.
- Complete translation of the Suyama construction.
- First experiments showed a speed-up of about 7 %.
- (See Cryptology ePrint Archive Report 2008/016 for details.)

Thank you for your attention!